Linked partition ideals and a family of quadruple summations

George E. Andrews and Shane Chern

Abstract. Recently, 4-regular partitions into distinct parts are connected with a family of overpartitions. In this paper, we provide a uniform extension of two relations due to Andrews for the two types of partitions. Such an extension is made possible with recourse to a new trivariate Rogers–Ramanujan type identity, which concerns a family of quadruple summations appearing as generating functions for the aforementioned overpartitions. More interestingly, the derivation of this Rogers–Ramanujan type identity is relevant to a certain well-poised basic hypergeometric series.

Keywords. Linked partition ideals, overpartitions, 4-regular partitions, generating functions, Andrews–Gordon type series, Rogers–Ramanujan type identities.

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1. Introduction

In the theory of basic hypergeometric series and integer partitions, the two Rogers–Ramanujan identities play an irreplaceable role. From an analytic perspective, they are

$$\prod_{n\geq 0}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}=\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n}, \tag{1.1}$$

$$\prod_{n\geq 0}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}=\sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n}. \tag{1.2}$$

Here and throughout we adopt the q-Pochhammer symbols for $n\in\mathbb{N}\cup\{\infty\}$,

$$(A;q)_n:=\prod_{k=0}^{n-1}(1-Aq^k)$$

and

$$(A_1,\ldots,A_\ell;q)_n:=(A_1;q)_n\cdots(A_\ell;q)_n.$$

In terms of integer partitions, the two identities may be interpreted as follows.

Theorem RR. (i) *The number of partitions of n into parts congruent to ±1 modulo 5 is the same as the number of partitions of n such that every two consecutive parts have difference at least 2.*

(ii) *The number of partitions of n into parts congruent to ±2 modulo 5 is the same as the number of partitions of n such that every two consecutive parts have difference at least 2 and that the smallest part is greater than 1.*

Since the first discovery of (1.1) and (1.2) by Rogers [20], which were overlooked for nearly two decades until Ramanujan [17,18] and Schur [21] independently reproduced them, there have been numerous generalizations and analogs of the Rogers–Ramanujan identities, among which Gordon's extension [15] to higher moduli is of substantial significance. Subsequently, Andrews [3] established the analytic counterpart of Gordon's result, namely, for $1 \leq i \leq k$ and $k \geq 2$,

$$\prod_{\substack{n\geq 1\\ n\not\equiv 0,\pm i \ (\mathrm{mod}\ 2k+1)}} \frac{1}{1-q^n} = \sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}}, \tag{1.3}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$. Summarizing from the right-hand side of the above, we are led to a family of q-multi-summations now known as the *series of Andrews–Gordon type*:

$$\sum_{n_1,\ldots,n_r\geq 0} \frac{(-1)^{L_1(n_1,\ldots,n_r)} q^{Q(n_1,\ldots,n_r)+L_2(n_1,\ldots,n_r)}}{(q^{A_1};q^{A_1})_{n_1}\cdots(q^{A_r};q^{A_r})_{n_r}}, \tag{1.4}$$

in which L_1 and L_2 are linear forms and Q is a quadratic form in the indices $n_1,\ldots,n_r$. It is usually expected to construct Andrews–Gordon type series or summations of alike shapes so that they are equal to a certain infinite product. Along this line, our first object is the following trivariate relation.

Theorem 1.1. *We have*

$$\begin{aligned}&(-xq;q^2)_\infty(-yq^2;q^4)_\infty\\ &\quad= \sum_{n_1,n_2,n_3,n_4\geq 0} \frac{x^{n_1+n_2+2n_4} y^{n_2+n_3} q^{n_1+3n_2+2n_3+4n_4}(1+x^2yq^{6+8(n_1+n_2+n_3+n_4)})}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^4;q^4)_{n_3}(q^4;q^4)_{n_4}}\\ &\qquad\times q^{4\binom{n_1}{2}+6\binom{n_2}{2}+4\binom{n_3}{2}+8\binom{n_4}{2}+4n_1n_2+4n_1n_3+4n_1n_4+4n_2n_3+4n_2n_4+4n_3n_4}.\end{aligned} \tag{1.5}$$

This identity is mainly motivated by a recent work of Andrews [7] on 4-regular partitions into distinct parts; here a partition is *k-regular* if no part is divisible by k. Owing to a theorem of Glaisher [14], such partitions are also equinumerous with partitions with no part appearing k or more times and this definition is often used in representation theory [16, p. 251]. In [7], Andrews connected 4-regular partitions into distinct parts with overpartitions that were introduced by Corteel and Lovejoy [13]. Recall that an *overpartition* of n is a partition of n where the first occurrence of each distinct part may be overlined. For example, 4 has fourteen overpartitions:

$$4,\ \overline{4},\ 3+1,\ \overline{3}+1,\ 3+\overline{1},\ \overline{3}+\overline{1},\ 2+2,\ \overline{2}+2,$$
$$2+1+1,\ \overline{2}+1+1,\ 2+\overline{1}+1,\ \overline{2}+\overline{1}+1,\ 1+1+1+1,\ \overline{1}+1+1+1.$$

Now consider the set $\mathscr{A}^{\vee}_{\{\overline{1}\}}$ of overpartitions such that

(1) Only odd parts **larger than 1** may be overlined;
(2) The difference between any two parts is ≥ 4 and the inequality is strict if the larger one is overlined or divisible by 4 with **the exception that $\overline{5}$ and 1 may simultaneously appear as parts**.

Andrews proved the following two results.

Theorem A1. *Let $A_1(n,m)$ count the number of overpartitions of n in $\mathscr{A}^{\vee}_{\{\overline{1}\}}$ into m parts with overlined parts and parts divisible by* 4 *counted with weight* 2*. Further,* *let $B_1(n,m)$ count the number of partitions into m distinct parts none divisible by 4. Then*

$$A_1(n,m) = B_1(n,m).$$

Theorem A2. *Let $A_2(n,m)$ count the number of overpartitions of n in $\mathscr{A}^{\vee}_{\{\overline{1}\}}$ into m parts with overlined parts counted with weight 3 and even parts counted with weight 2. Further, let $B_2(n,m)$ count the number of partitions into m odd parts none appearing more than three times. Then*

$$A_2(n,m) = B_2(n,m).$$

Now note that

$$\sum_{m,n\geq 0} B_1(n,m)x^m q^n = \prod_{\substack{k\geq 1\\ k\not\equiv 0 \pmod 4}} (1+xq^k)$$
$$= (-xq;q^2)_\infty(-xq^2;q^4)_\infty.$$

Meanwhile,

$$\sum_{m,n\geq 0} B_2(n,m)x^m q^n = \prod_{k\geq 1}(1+xq^{2k-1}+x^2q^{2(2k-1)}+x^3q^{3(2k-1)})$$
$$= (-xq;q^2)_\infty(-x^2q^2;q^4)_\infty.$$

Hence the two infinite products are special cases of the left-hand side of (1.5). Naturally, it is then expected that the right-hand side of (1.5) should characterize the overpartition set $\mathscr{A}^{\vee}_{\{\overline{1}\}}$.

For this purpose, we first loosen the conditions for $\mathscr{A}^{\vee}_{\{\overline{1}\}}$.

Definition 1.1. Let $\mathscr{A}$ denote the set of overpartitions such that

(1) Only odd parts may be overlined;
(2) The difference between any two parts is ≥ 4 and the inequality is strict if the larger one is overlined or divisible by 4.

Our next object is to establish quinvariate generating function formulas for the above overpartitions, possibly with extra restrictions on the smallest part, such as

$$\sum_{\lambda\in\mathscr{A}} x^{\sharp(\lambda)}y_1^{\sharp_{2,4}(\lambda)}y_2^{\sharp_{0,4}(\lambda)}z^{\mathsf{O}(\lambda)}q^{|\lambda|}.$$

Here we adopt the notations that for any (over)partition λ, $|\lambda|$ and $\sharp(\lambda)$ are the sum of all parts (namely, the *size*) and the number of parts (namely, the *length*) in λ, respectively, and $\sharp_{a,M}(\lambda)$ is the number of parts in λ that are congruent to a modulo M. Meanwhile, we denote by $\mathsf{O}(\lambda)$ the number of overlined parts in an overpartition λ.

For the sake of brevity, we postpone the presentation of these generating functions until Theorem 5.1. However, we state here that Theorems A1 and A2 may be unified with an additional parameter introduced.

Theorem 1.2. *Let $A(n,m,\ell)$ count the number of overpartitions λ of n in $\mathscr{A}^{\vee}_{\{\overline{1}\}}$ such that $\sharp_{1,2}(\lambda)+2\sharp_{0,4}(\lambda) = m$ and $\sharp_{2,4}(\lambda)+\mathsf{O}(\lambda) = \ell$. Further, let $B(n,m,\ell)$*

count the number of 4*-regular partitions into distinct parts with* m *odd parts and* ℓ *even parts. Then*

$$A(n,m,\ell)=B(n,m,\ell). \tag{1.6}$$

2. A trivariate identity

To establish Theorem 1.1, we require the following trivariate relation, which is of independent interest.

Theorem 2.1. *We have*

$$\frac{(-x;q)_\infty(xy;q)_\infty}{(x^2yq^2;q^2)_\infty}=\sum_{n\geq 0}\frac{x^nq^{\binom{n}{2}}(1-x^2y^2q^{4n})(xy;q)_n(y;q^2)_n}{(q;q)_n(x^2yq^2;q^2)_n}. \tag{2.1}$$

For its proof, we recall that Andrews introduced in [1] a family of q-series arising from a certain well-poised basic hypergeometric series:

$$\begin{aligned}&H_{k,i}(a_1,a_2,a_3;x,q)\\&\quad:=\frac{(\frac{xq}{a_1},\frac{xq}{a_2},\frac{xq}{a_3};q)_\infty}{(xq;q)_\infty}\sum_{n\geq 0}\frac{\left(\frac{x^k}{a_1a_2a_3}\right)^nq^{(k-1)n^2+(2-i)n}(1-x^iq^{2ni})(x,a_1,a_2,a_3;q)_n}{(1-x)(q,\frac{xq}{a_1},\frac{xq}{a_2},\frac{xq}{a_3};q)_n}.\end{aligned}$$

From [1, p. 439, Eq. (3.7)],

$$H_{1,1}(a_1,a_2,a_3;x,q)=\frac{(\frac{xq}{a_1a_2},\frac{xq}{a_2a_3},\frac{xq}{a_3a_1};q)_\infty}{(\frac{xq}{a_1a_2a_3};q)_\infty}. \tag{2.2}$$

Also, [1, p. 439, Eq. (3.4)] tells us that the following q-difference equation is valid:

$$\begin{aligned}H_{1,2}(a_1,a_2,a_3;xq,q)&=H_{1,1}(a_1,a_2,a_3;x,q)\\&\quad+xq(\sigma_1-xq\sigma_3)H_{1,1}(a_1,a_2,a_3;xq,q),\end{aligned} \tag{2.3}$$

where $\sigma_j=\sigma_j(a_1^{-1},a_2^{-1},a_3^{-1})$ is the j-th elementary symmetric function of a_1^{-1}, a_2^{-1} and a_3^{-1}.

Proof. Define

$$h(a_1,a_2;x,q):=\lim_{a_3\to\infty}H_{1,2}(a_1,a_2,a_3;x,q).$$

Then

$$h(a_1,a_2;x,q)=\frac{(\frac{xq}{a_1},\frac{xq}{a_2};q)_\infty}{(xq;q)_\infty}\sum_{n\geq 0}\frac{\left(\frac{x}{a_1a_2}\right)^n(-1)^nq^{\binom{n}{2}}(1-x^2q^{4n})(x,a_1,a_2;q)_n}{(1-x)(q,\frac{xq}{a_1},\frac{xq}{a_2};q)_n}.$$

Now taking $(x,a_1,a_2)\mapsto(xy,y^{1/2},-y^{1/2})$ gives

$$\sum_{n\geq 0}\frac{x^nq^{\binom{n}{2}}(1-x^2y^2q^{4n})(xy;q)_n(y;q^2)_n}{(q;q)_n(x^2yq^2;q^2)_n}=\frac{(xy;q)_\infty}{(x^2yq^2;q^2)_\infty}h(y^{1/2},-y^{1/2};xy,q).$$

Meanwhile, it is known from (2.2) and (2.3) that

$$\begin{aligned}H_{1,2}(a_1,a_2,a_3;x,q)&=\frac{(\frac{x}{a_1a_2},\frac{x}{a_2a_3},\frac{x}{a_3a_1};q)_\infty}{(\frac{x}{a_1a_2a_3};q)_\infty}\\&\quad+x\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}-\frac{x}{a_1a_2a_3}\right)\frac{(\frac{xq}{a_1a_2},\frac{xq}{a_2a_3},\frac{xq}{a_3a_1};q)_\infty}{(\frac{xq}{a_1a_2a_3};q)_\infty}.\end{aligned}$$

We still let $a_3 \to \infty$ and then take $(x, a_1, a_2) \mapsto (xy, y^{1/2}, -y^{1/2})$. Thus,

$$h(y^{1/2}, -y^{1/2}; xy, q) = (-x; q)_\infty.$$

Substituting the above into the previous relation gives the required identity. □

3. Proof of Theorem 1.1

We start with a list of well-known relations for basic hypergeometric series.

▷ Euler's first sum [6, Eq. (2.2.5)]:

$$\sum_{n\ge 0} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}. \tag{3.1}$$

▷ Euler's second sum [6, Eq. (2.2.6)]:

$$\sum_{n\ge 0} \frac{z^n q^{\binom{n}{2}}}{(q;q)_n} = (-z;q)_\infty. \tag{3.2}$$

▷ The q-binomial theorem [6, Eq. (2.2.1)]:

$$\sum_{n\ge 0} \frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{3.3}$$

Now let us establish an equivalent identity of (1.5).

Theorem 3.1. *We have*

$$\begin{aligned}\frac{1}{(xq;q^2)_\infty (yq^2;q^4)_\infty} &= \sum_{n_1,n_2,n_3,n_4\ge 0} \frac{x^{n_1+n_2+2n_4} y^{n_2+n_3} q^{n_1+3n_2+2n_3+2n_4}(1+x^2yq^{4+4(n_1+n_3)})}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^4;q^4)_{n_3}(q^4;q^4)_{n_4}}\\ &\quad \times q^{2\binom{n_1}{2}+2n_1n_2+4n_1n_3+4n_3n_4}. \end{aligned}\tag{3.4}$$

Proof. We first consider the inner summation over n_4 and obtain by (3.1) that

$$\begin{aligned}&\sum_{n_1,n_2,n_3,n_4\ge 0} \frac{x^{n_1+n_2+2n_4} y^{n_2+n_3} q^{2\binom{n_1}{2}+2n_1n_2+4n_1n_3+4n_3n_4+n_1+3n_2+2n_3+2n_4}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^4;q^4)_{n_3}(q^4;q^4)_{n_4}}\\
&= \sum_{n_1,n_2,n_3\ge 0} \frac{x^{n_1+n_2} y^{n_2+n_3} q^{2\binom{n_1}{2}+2n_1n_2+4n_1n_3+n_1+3n_2+2n_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^4;q^4)_{n_3}} \sum_{n_4\ge 0}\frac{(x^2q^{4n_3+2})^{n_4}}{(q^4;q^4)_{n_4}}\\
&= \sum_{n_1,n_2,n_3\ge 0} \frac{x^{n_1+n_2} y^{n_2+n_3} q^{2\binom{n_1}{2}+2n_1n_2+4n_1n_3+n_1+3n_2+2n_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^4;q^4)_{n_3}(x^2q^{4n_3+2};q^4)_\infty}\\
&= \frac{1}{(x^2q^2;q^4)_\infty}\sum_{n_1,n_2,n_3\ge 0} \frac{x^{n_1+n_2} y^{n_2+n_3} q^{2\binom{n_1}{2}+2n_1n_2+4n_1n_3+n_1+3n_2+2n_3}(x^2q^2;q^4)_{n_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^4;q^4)_{n_3}}.\end{aligned}$$

Now we further work on the inner summations over n_2 and n_3, respectively, with the application of (3.1) and (3.3), and find that

$$\sum_{n_1,n_2,n_3,n_4\ge 0} \frac{x^{n_1+n_2+2n_4} y^{n_2+n_3} q^{2\binom{n_1}{2}+2n_1n_2+4n_1n_3+4n_3n_4+n_1+3n_2+2n_3+2n_4}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^4;q^4)_{n_3}(q^4;q^4)_{n_4}}$$

$$= \frac{1}{(x^2q^2;q^4)_\infty} \sum_{n_1\ge 0} \frac{x^{n_1}q^{2\binom{n_1}{2}+n_1}}{(q^2;q^2)_{n_1}} \sum_{n_2\ge 0} \frac{(xyq^{2n_1+3})^{n_2}}{(q^2;q^2)_{n_2}} \sum_{n_3\ge 0} \frac{(yq^{4n_1+2})^{n_3}(x^2q^2;q^4)_{n_3}}{(q^4;q^4)_{n_3}}$$

$$= \frac{1}{(x^2q^2;q^4)_\infty} \sum_{n_1\ge 0} \frac{x^{n_1}q^{2\binom{n_1}{2}+n_1}(x^2yq^{4n_1+4};q^4)_\infty}{(q^2;q^2)_{n_1}(xyq^{2n_1+3};q^2)_\infty(yq^{4n_1+2};q^4)_\infty}$$

$$= \frac{(x^2yq^4;q^4)_\infty}{(x^2q^2;q^4)_\infty(xyq^3;q^2)_\infty(yq^2;q^4)_\infty} \sum_{n_1\ge 0} \frac{x^{n_1}q^{2\binom{n_1}{2}+n_1}(xyq^3;q^2)_{n_1}(yq^2;q^4)_{n_1}}{(q^2;q^2)_{n_1}(x^2yq^4;q^4)_{n_1}}.$$

Similarly,

$$\sum_{n_1,n_2,n_3,n_4\ge 0} \frac{x^{n_1+n_2+2n_4+2}y^{n_2+n_3+1}q^{2\binom{n_1}{2}+2n_1n_2+4n_1n_3+4n_3n_4+5n_1+3n_2+6n_3+2n_4+4}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^4;q^4)_{n_3}(q^4;q^4)_{n_4}}$$

$$= \frac{x^2yq^4(x^2yq^8;q^4)_\infty}{(x^2q^2;q^4)_\infty(xyq^3;q^2)_\infty(yq^6;q^4)_\infty} \sum_{n_1\ge 0} \frac{x^{n_1}q^{2\binom{n_1}{2}+5n_1}(xyq^3;q^2)_{n_1}(yq^6;q^4)_{n_1}}{(q^2;q^2)_{n_1}(x^2yq^8;q^4)_{n_1}}.$$

Thus,

$$\begin{aligned}
&\mathrm{RHS}\,(3.4)\\
&= \frac{(x^2yq^4;q^4)_\infty}{(x^2q^2;q^4)_\infty(xyq^3;q^2)_\infty(yq^2;q^4)_\infty}\\
&\quad \times \sum_{n_1\ge 0} \frac{x^{n_1}q^{2\binom{n_1}{2}+n_1}(xyq^3;q^2)_{n_1}(yq^2;q^4)_{n_1}}{(q^2;q^2)_{n_1}(x^2yq^4;q^4)_{n_1}} \left(1+\frac{x^2yq^{4n_1+4}(1-yq^{4n_1+2})}{1-x^2yq^{4n_1+4}}\right)\\
&= \frac{(x^2yq^4;q^4)_\infty}{(x^2q^2;q^4)_\infty(xyq^3;q^2)_\infty(yq^2;q^4)_\infty}\\
&\quad \times \sum_{n_1\ge 0} \frac{x^{n_1}q^{2\binom{n_1}{2}+n_1}(xyq^3;q^2)_{n_1}(yq^2;q^4)_{n_1}}{(q^2;q^2)_{n_1}(x^2yq^4;q^4)_{n_1}} \cdot \frac{1-x^2y^2q^{8n_1+6}}{1-x^2yq^{4n_1+4}}\\
&= \frac{(x^2yq^8;q^4)_\infty}{(x^2q^2;q^4)_\infty(xyq^3;q^2)_\infty(yq^2;q^4)_\infty}\\
&\quad \times \sum_{n_1\ge 0} \frac{x^{n_1}q^{2\binom{n_1}{2}+n_1}(1-x^2y^2q^{8n_1+6})(xyq^3;q^2)_{n_1}(yq^2;q^4)_{n_1}}{(q^2;q^2)_{n_1}(x^2yq^8;q^4)_{n_1}}.
\end{aligned}$$

Finally, in (2.1), we take $(x,y,q)\mapsto(xq,yq^2,q^2)$. Then

$$\begin{aligned}
\mathrm{RHS}\,(3.4) &= \frac{(x^2yq^8;q^4)_\infty}{(x^2q^2;q^4)_\infty(xyq^3;q^2)_\infty(yq^2;q^4)_\infty} \cdot \frac{(-xq;q^2)_\infty(xyq^3;q^2)_\infty}{(x^2yq^8;q^4)_\infty}\\
&= \frac{1}{(xq;q^2)_\infty(yq^2;q^4)_\infty},
\end{aligned}$$

as required. □

To see why (3.4) and (1.5) are equivalent, we need a functional operator $\mathcal{B}$ defined on $\mathbb{C}[[q]][[x,y]]$ by

$$\mathcal{B}\left(\sum_{m,n\ge 0} c_{m,n}x^my^n\right) := \sum_{m,n\ge 0} c_{m,n}q^{2\binom{m}{2}+4\binom{n}{2}}x^my^n,$$

where the coefficients $c_{m,n}$ are in $\mathbb{C}[[q]]$. This operator can be treated as a specialization of the q-Borel operators [11, 19, 22].

Proof of (1.5) *from* (3.4). Note that

$$\begin{aligned}
&\mathcal{B}\big(\operatorname{RHS}(3.4)\big)\\
&= \sum_{n_1,n_2,n_3,n_4\ge 0} \frac{x^{n_1+n_2+2n_4} y^{n_2+n_3} q^{2\binom{n_1}{2}+2n_1n_2+4n_1n_3+4n_3n_4+n_1+3n_2+2n_3+2n_4}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^4;q^4)_{n_3}(q^4;q^4)_{n_4}}\\
&\quad \times \Big(q^{2\binom{n_1+n_2+2n_4}{2}+4\binom{n_2+n_3}{2}} + x^2yq^{2\binom{n_1+n_2+2n_4+2}{2}+4\binom{n_2+n_3+1}{2}+4+4(n_1+n_3)}\Big)\\
&= \sum_{n_1,n_2,n_3,n_4\ge 0} \frac{x^{n_1+n_2+2n_4} y^{n_2+n_3} q^{n_1+3n_2+2n_3+4n_4}\big(1+x^2yq^{6+8(n_1+n_2+n_3+n_4)}\big)}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^4;q^4)_{n_3}(q^4;q^4)_{n_4}}\\
&\quad \times q^{4\binom{n_1}{2}+6\binom{n_2}{2}+4\binom{n_3}{2}+8\binom{n_4}{2}+4n_1n_2+4n_1n_3+4n_1n_4+4n_2n_3+4n_2n_4+4n_3n_4},
\end{aligned}$$

which is exactly the right-hand side of (1.5). On the other hand, we rewrite the left-hand side of (3.4) in light of (3.1),

$$\operatorname{LHS}(3.4) = \sum_{m_1,m_2\ge 0} \frac{x^{m_1}y^{m_2}q^{m_1+2m_2}}{(q^2;q^2)_{m_1}(q^4;q^4)_{m_2}}.$$

Hence,

$$\begin{aligned}
\mathcal{B}\big(\operatorname{LHS}(3.4)\big) &= \sum_{m_1,m_2\ge 0} \frac{x^{m_1}y^{m_2}q^{2\binom{m_1}{2}+4\binom{m_2}{2}+m_1+2m_2}}{(q^2;q^2)_{m_1}(q^4;q^4)_{m_2}}\\
&= (-xq;q^2)_\infty(-yq^2;q^4)_\infty,
\end{aligned}$$

where (3.2) is applied. Finally,

$$\operatorname{RHS}(1.5) = \mathcal{B}\big(\operatorname{RHS}(3.4)\big) = \mathcal{B}\big(\operatorname{LHS}(3.4)\big) = \operatorname{LHS}(1.5),$$

as desired. □

4. Span one linked partition ideals

Now we shall consider the generating functions related to the overpartitions in $\mathscr{A}$. For this purpose, we take advantage of the framework of *span one linked partition ideals* introduced by Andrews [2, 4, 5] in the 1970s and reflourished in a series of recent projects mainly led by Chern [8–12]. It is necessary to point out that linked partition ideals are originally considered over ordinary partitions; see, for instance, [6, Chapter 8] or [8, Definition 2.1]. However, according to the generic setting introduced in [9], including **overpartitions** will *not* bring about any extra issue.

Definition 4.1. Assume that we are given

- a finite set $\Pi = \{\pi_1, \pi_2, \dots, \pi_K\}$ of **overpartitions** with $\pi_1 = \varnothing$, the empty partition,
- a *map of linking sets*, $\mathcal{L} : \Pi \to P(\Pi)$, the power set of Π, with especially, $\mathcal{L}(\pi_1) = \mathcal{L}(\varnothing) = \Pi$ and $\pi_1 = \varnothing \in \mathcal{L}(\pi_k)$ for any $1 \le k \le K$,
- and a positive integer T, called the *modulus*, which is greater than or equal to the largest part among all **overpartitions** in Π.

We say a *span one linked partition ideal* $\mathscr{I} = \mathscr{I}(\langle \Pi, \mathcal{L}\rangle, T)$ is the collection of all **overpartitions** of the form

$$\begin{aligned}\lambda &= \phi^0(\lambda_0) \oplus \phi^T(\lambda_1) \oplus \cdots \oplus \phi^{NT}(\lambda_N) \oplus \phi^{(N+1)T}(\pi_1) \oplus \phi^{(N+2)T}(\pi_1) \oplus \cdots \\ &= \phi^0(\lambda_0) \oplus \phi^T(\lambda_1) \oplus \cdots \oplus \phi^{NT}(\lambda_N), \end{aligned} \tag{4.1}$$

where $\lambda_i \in \mathcal{L}(\lambda_{i-1})$ for each i and λ_N is not the empty partition. We also include in $\mathscr{I}$ the empty partition, which corresponds to $\phi^0(\pi_1) \oplus \phi^T(\pi_1) \oplus \cdots$. Here for any two **overpartitions** μ and ν, $\mu \oplus \nu$ gives an **overpartition** by collecting all parts in μ and ν, and $\phi^m(\mu)$ gives an **overpartition** by adding m to each part of μ with **overlines preserved**.

Recall that $\mathscr{A}$ denotes the set of overpartitions such that

(1) Only odd parts may be overlined;
(2) The difference between any two parts is ≥ 4 and the inequality is strict if the larger one is overlined or divisible by 4.

Lemma 4.1. *$\mathscr{A}$ equals the span one linked partition ideal $\mathscr{I}(\langle \Pi, \mathcal{L}\rangle, 4)$, where $\Pi = \{\pi_1 = \varnothing, \pi_2 = (1), \pi_3 = (\overline{1}), \pi_4 = (2), \pi_5 = (3), \pi_6 = (\overline{3}), \pi_7 = (4)\}$ and*

$$\begin{cases} \mathcal{L}(\pi_1) = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7\}, \\ \mathcal{L}(\pi_2) = \mathcal{L}(\pi_3) = \{\pi_1, \pi_2, \pi_4, \pi_5, \pi_6, \pi_7\}, \\ \mathcal{L}(\pi_4) = \{\pi_1, \pi_4, \pi_5, \pi_6, \pi_7\}, \\ \mathcal{L}(\pi_5) = \mathcal{L}(\pi_6) = \{\pi_1, \pi_5, \pi_7\}, \\ \mathcal{L}(\pi_7) = \{\pi_1\}. \end{cases}$$

Proof. It is clear that all overpartitions in $\mathscr{I}(\langle \Pi, \mathcal{L}\rangle, 4)$ satisfy the conditions for $\mathscr{A}$. For the other direction, we decompose each overpartition in $\mathscr{A}$ into blocks $\mathbf{B}_0, \mathbf{B}_1, \ldots$ such that all parts (including those that are overlined) between $4i+1$ and $4i+4$ fall into block $\mathbf{B}_i$. It is plain that $\phi^{-4i}(\mathbf{B}_i)$ is exclusively from Π. Further, if $\phi^{-4i}(\mathbf{B}_i)$ is π_1 so that $\mathbf{B}_i$ is $\varnothing$, then $\phi^{-4(i+1)}(\mathbf{B}_{i+1})$ can be any among Π. If $\phi^{-4i}(\mathbf{B}_i)$ is π_2 or π_3 so that $\mathbf{B}_i$ is $(4i+1)$ or $(\overline{4i+1})$, then $\mathbf{B}_{i+1}$ cannot be $(\overline{4i+5})$ by the second condition for $\mathscr{A}$ so that $\phi^{-4(i+1)}(\mathbf{B}_{i+1})$ cannot be π_3. One may carry out similar arguments for other possibilities of $\phi^{-4i}(\mathbf{B}_i)$ and the details are omitted. ☐

Example 4.1. As in (4.1), we decompose the overpartition $\overline{1}+8+14+\overline{19}+23+27$ by

$$\phi^0(\overline{1}) \oplus \phi^4(4) \oplus \phi^8(\varnothing) \oplus \phi^{12}(2) \oplus \phi^{16}(\overline{3}) \oplus \phi^{20}(3) \oplus \phi^{24}(3),$$

which corresponds to the chain $\pi_3\pi_7\pi_1\pi_4\pi_6\pi_5\pi_5\pi_1\pi_1\cdots$.

Throughout, we always decompose overpartitions $\lambda \in \mathscr{A} = \mathscr{I}(\langle \Pi, \mathcal{L}\rangle, 4)$ as in (4.1). Now define for $1 \le k \le 7$:

$$G_k(x) = G_k(x, y_1, y_2, z, q) := \sum_{\substack{\lambda \in \mathscr{A} \\ \lambda_0 = \pi_k}} x^{\sharp(\lambda)} y_1^{\sharp_{2,4}(\lambda)} y_2^{\sharp_{0,4}(\lambda)} z^{\mathsf{O}(\lambda)} q^{|\lambda|}. \tag{4.2}$$

In other words, $G_k(x)$ is the generating function for overpartitions in $\mathscr{A}$ whose first decomposed block $\mathbf{B}_0$ equals π_k. From the above construction, it is plain that

$$G_k(x) = x^{\sharp(\pi_k)} y_1^{\sharp_{2,4}(\pi_k)} y_2^{\sharp_{0,4}(\pi_k)} z^{\mathsf{O}(\pi_k)} q^{|\pi_k|} \sum_{j:\pi_j \in \mathcal{L}(\pi_k)} G_j(xq^4).$$

Hence,

$$\begin{pmatrix} G_1(x) \\ G_2(x) \\ \vdots \\ G_7(x) \end{pmatrix} = \mathbf{W}.\mathbf{A}.\begin{pmatrix} G_1(xq^4) \\ G_2(xq^4) \\ \vdots \\ G_7(xq^4) \end{pmatrix}, \tag{4.3}$$

where

$$\mathbf{W} = \operatorname{diag}(1, xq, xzq, xy_1q^2, xq^3, xzq^3, xy_2q^4)$$

and

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We further write

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_7(x) \end{pmatrix} = \mathbf{A}.\begin{pmatrix} G_1(x) \\ G_2(x) \\ \vdots \\ G_7(x) \end{pmatrix}. \tag{4.4}$$

Then

$$\begin{pmatrix} F_1(x) \\ F_2(x) \\ \vdots \\ F_7(x) \end{pmatrix} = \mathbf{A}.\mathbf{W}.\begin{pmatrix} F_1(xq^4) \\ F_2(xq^4) \\ \vdots \\ F_7(xq^4) \end{pmatrix}. \tag{4.5}$$

5. Quinvariate generating functions

Here our object is to establish related generating functions for $\mathscr{A}$. Letting S be a collection of parts, we denote by $\mathscr{A}_S$ the subset of overpartitions in $\mathscr{A}$ such that parts from S are forbidden.

Theorem 5.1. *We have*

$$\sum_{\lambda\in\mathscr{A}} x^{\sharp(\lambda)} y_1^{\sharp_{2,4}(\lambda)} y_2^{\sharp_{0,4}(\lambda)} z^{\mathsf{O}(\lambda)} q^{|\lambda|}$$
$$= \sum_{n_1,n_2,n_3,n_4\ge 0} \frac{x^{n_1+n_2+n_3+n_4} y_1^{n_3} y_2^{n_4} z^{n_2} q^{n_1+n_2+2n_3+4n_4}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^4;q^4)_{n_3}(q^4;q^4)_{n_4}}$$
$$\times q^{4\binom{n_1}{2}+6\binom{n_2}{2}+4\binom{n_3}{2}+8\binom{n_4}{2}+4n_1n_2+4n_1n_3+4n_1n_4+4n_2n_3+4n_2n_4+4n_3n_4}, \tag{5.1}$$

$$\sum_{\lambda\in\mathscr{A}_{\{\overline{1}\}}} x^{\sharp(\lambda)} y_1^{\sharp_{2,4}(\lambda)} y_2^{\sharp_{0,4}(\lambda)} z^{\mathsf{O}(\lambda)} q^{|\lambda|}$$
$$= \sum_{n_1,n_2,n_3,n_4\ge 0} \frac{x^{n_1+n_2+n_3+n_4} y_1^{n_3} y_2^{n_4} z^{n_2} q^{n_1+3n_2+2n_3+4n_4}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^4;q^4)_{n_3}(q^4;q^4)_{n_4}}$$
$$\times q^{4\binom{n_1}{2}+6\binom{n_2}{2}+4\binom{n_3}{2}+8\binom{n_4}{2}+4n_1n_2+4n_1n_3+4n_1n_4+4n_2n_3+4n_2n_4+4n_3n_4}, \tag{5.2}$$

$$\sum_{\lambda\in\mathscr{A}_{\{1,\overline{1}\}}} x^{\sharp(\lambda)}y_1^{\sharp_{2,4}(\lambda)}y_2^{\sharp_{0,4}(\lambda)}z^{\mathsf{O}(\lambda)}q^{|\lambda|}$$
$$= \sum_{n_1,n_2,n_3,n_4\geq 0} \frac{x^{n_1+n_2+n_3+n_4}y_1^{n_3}y_2^{n_4}z^{n_2}q^{3n_1+3n_2+2n_3+4n_4}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^4;q^4)_{n_3}(q^4;q^4)_{n_4}}$$
$$\times q^{4\binom{n_1}{2}+6\binom{n_2}{2}+4\binom{n_3}{2}+8\binom{n_4}{2}+4n_1n_2+4n_1n_3+4n_1n_4+4n_2n_3+4n_2n_4+4n_3n_4}, \quad (5.3)$$

$$\sum_{\lambda\in\mathscr{A}_{\{1,\overline{1},2,\overline{3}\}}} x^{\sharp(\lambda)}y_1^{\sharp_{2,4}(\lambda)}y_2^{\sharp_{0,4}(\lambda)}z^{\mathsf{O}(\lambda)}q^{|\lambda|}$$
$$= \sum_{n_1,n_2,n_3,n_4\geq 0} \frac{x^{n_1+n_2+n_3+n_4}y_1^{n_3}y_2^{n_4}z^{n_2}q^{3n_1+5n_2+6n_3+4n_4}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^4;q^4)_{n_3}(q^4;q^4)_{n_4}}$$
$$\times q^{4\binom{n_1}{2}+6\binom{n_2}{2}+4\binom{n_3}{2}+8\binom{n_4}{2}+4n_1n_2+4n_1n_3+4n_1n_4+4n_2n_3+4n_2n_4+4n_3n_4}. \quad (5.4)$$

To begin with, we note that

$$\sum_{\lambda\in\mathscr{A}} x^{\sharp(\lambda)}y_1^{\sharp_{2,4}(\lambda)}y_2^{\sharp_{0,4}(\lambda)}z^{\mathsf{O}(\lambda)}q^{|\lambda|} = \sum_{k\in\{1,2,3,4,5,6,7\}} G_k(x) = F_1(x),$$
$$\sum_{\lambda\in\mathscr{A}_{\{\overline{1}\}}} x^{\sharp(\lambda)}y_1^{\sharp_{2,4}(\lambda)}y_2^{\sharp_{0,4}(\lambda)}z^{\mathsf{O}(\lambda)}q^{|\lambda|} = \sum_{k\in\{1,2,4,5,6,7\}} G_k(x) = F_2(x) = F_3(x),$$
$$\sum_{\lambda\in\mathscr{A}_{\{1,\overline{1}\}}} x^{\sharp(\lambda)}y_1^{\sharp_{2,4}(\lambda)}y_2^{\sharp_{0,4}(\lambda)}z^{\mathsf{O}(\lambda)}q^{|\lambda|} = \sum_{k\in\{1,4,5,6,7\}} G_k(x) = F_4(x),$$
$$\sum_{\lambda\in\mathscr{A}_{\{1,\overline{1},2,\overline{3}\}}} x^{\sharp(\lambda)}y_1^{\sharp_{2,4}(\lambda)}y_2^{\sharp_{0,4}(\lambda)}z^{\mathsf{O}(\lambda)}q^{|\lambda|} = \sum_{k\in\{1,5,7\}} G_k(x) = F_5(x) = F_6(x).$$

Thus it suffices to determine the expression of each $F_k(x)$. If we treat (4.5) as a system of q-difference equations, its formal power series solution $\big(F_1(x),\ldots,F_7(x)\big)$ is uniquely determined by $\big(F_1(0),\ldots,F_7(0)\big)$. Further, according to our construction, $F_k(0)=1$ for each k.

Recall that in [9] and [10], a generic family of q-multi-summations was considered. Let R be a given positive integer and fix a symmetric matrix $\underline{\boldsymbol{\alpha}} = (\alpha_{i,j}) \in \mathrm{Mat}_{R\times R}(\mathbb{N})$ and a vector $\underline{\mathbf{A}} = (A_r) \in \mathbb{N}_{>0}^R$. Also fix J vectors $\underline{\boldsymbol{\gamma_j}} = (\gamma_{j,r}) \in \mathbb{N}_{\geq 0}^R$ for $j=1,2,\ldots,J$. Define for indeterminates $x_1,x_2,\ldots,x_J$ and q the following q-multi-summation $H(\underline{\boldsymbol{\beta}}) = H(\beta_1,\ldots,\beta_R)$ with $\underline{\boldsymbol{\beta}}\in\mathbb{Z}^R$:

$$H(\underline{\boldsymbol{\beta}}) = H(\beta_1,\ldots,\beta_R) := \sum_{n_1,\ldots,n_R\geq 0} \frac{x_1^{\sum_{r=1}^R \gamma_{1,r}n_r}\cdots x_J^{\sum_{r=1}^R \gamma_{J,r}n_r}}{(q^{A_1};q^{A_1})_{n_1}\cdots(q^{A_R};q^{A_R})_{n_R}}$$
$$\times q^{\sum_{r=1}^R \alpha_{r,r}\binom{n_r}{2}+\sum_{1\leq i<j\leq R}\alpha_{i,j}n_in_j+\sum_{r=1}^R\beta_r n_r}.$$

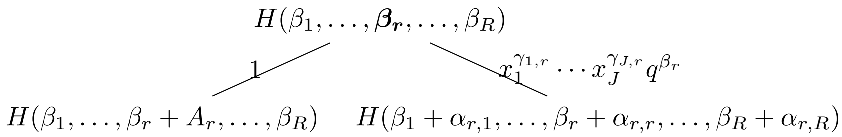
We require a recurrence for $H(\underline{\boldsymbol{\beta}})$ given in [10, Lemma 2.1].

Lemma 5.2. *For $1\leq r\leq R$, we have*

$$H(\beta_1,\ldots,\beta_r,\ldots,\beta_R) = H(\beta_1,\ldots,\beta_r+A_r,\ldots,\beta_R)$$
$$+ x_1^{\gamma_{1,r}}\cdots x_J^{\gamma_{J,r}}q^{\beta_r}H(\beta_1+\alpha_{r,1},\ldots,\beta_r+\alpha_{r,r},\ldots,\beta_R+\alpha_{r,R}).$$

As in [10], we illustrate the above relation by a binary tree with the coordinate β_r shown in boldface; see Figure 1.

FIGURE 1. Node $H(\beta_1, \ldots, \beta_r, \ldots, \beta_R)$ and its children



Now let us choose

$$\underline{\boldsymbol{\alpha}} = \begin{pmatrix} 4 & 4 & 4 & 4 \\ 4 & 6 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 8 \end{pmatrix}, \qquad \underline{\mathbf{A}} = (2, 2, 4, 4),$$

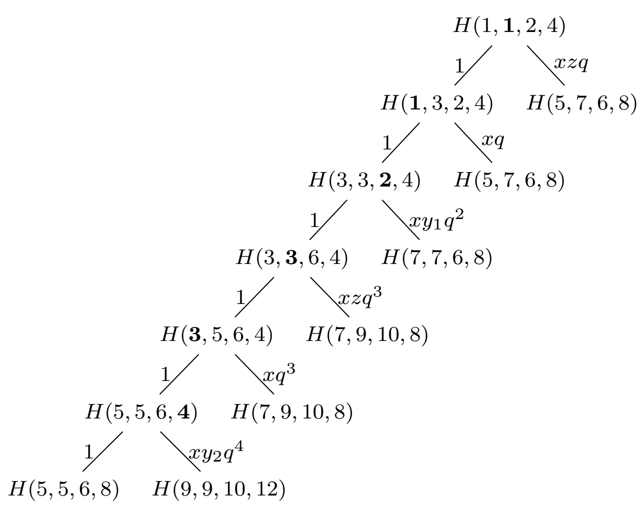
and

$$\begin{aligned} x_1 &= x, & \underline{\boldsymbol{\gamma_1}} &= (1, 1, 1, 1), \\ x_2 &= y_1, & \underline{\boldsymbol{\gamma_2}} &= (0, 0, 1, 0), \\ x_3 &= y_2, & \underline{\boldsymbol{\gamma_3}} &= (0, 0, 0, 1), \\ x_4 &= z, & \underline{\boldsymbol{\gamma_4}} &= (0, 1, 0, 0). \end{aligned}$$

To prove Theorem 5.1, it is sufficient to confirm that

$$\begin{pmatrix} H(1,1,2,4) \\ H(1,3,2,4) \\ H(1,3,2,4) \\ H(3,3,2,4) \\ H(3,5,6,4) \\ H(3,5,6,4) \\ H(5,5,6,8) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & & & & \\ & xq & & & & & \\ & & xzq & & & & \\ & & & xy_1q^2 & & & \\ & & & & xq^3 & & \\ & & & & & xzq^3 & \\ & & & & & & xy_2q^4 \end{pmatrix} \cdot \begin{pmatrix} H(5,5,6,8) \\ H(5,7,6,8) \\ H(5,7,6,8) \\ H(7,7,6,8) \\ H(7,9,10,8) \\ H(7,9,10,8) \\ H(9,9,10,12) \end{pmatrix}. \tag{5.5}$$

FIGURE 2. The binary tree for (5.5)



Proof. We make use of Lemma 5.2 and illustrate the proof by the binary tree in Figure 2. For instance, from the node $H(\mathbf{3},5,6,4)$ at the fifth level, we apply Lemma 5.2 to the first coordinate and obtain

$$H(\mathbf{3},5,6,4)=H(5,5,6,4)+xq^3H(7,9,10,8).$$

We further apply Lemma 5.2 to the fourth coordinate of $H(5,5,6,\mathbf{4})$ and obtain

$$H(5,5,6,\mathbf{4})=H(5,5,6,8)+xy_2q^4H(9,9,10,12).$$

Hence,

$$H(3,5,6,4)=H(5,5,6,8)+xq^3H(7,9,10,8)+xy_2q^4H(9,9,10,12),$$

thereby confirming the fifth and sixth rows of (5.5). Other rows can be argued in the same vein. □

6. Proof of Theorem 1.2

Theorem 1.2 is a direct consequence of (1.5) and (5.2). Recall that $\mathscr{A}^{\veebar}_{\{\overline{1}\}}$ denotes the set of overpartitions such that

(1) Only odd parts **larger than 1** may be overlined;
(2) The difference between any two parts is ≥ 4 and the inequality is strict if the larger one is overlined or divisible by 4 with **the exception that $\overline{5}$ and 1 may simultaneously appear as parts**.

Now there are two cases. **(i).** If $\overline{5}$ and 1 do not simultaneously appear as parts, then such overpartitions are exactly those in $\mathscr{A}_{\{\overline{1}\}}$. **(ii).** If $\overline{5}$ and 1 simultaneously appear as parts, then apart from them, the smallest part is at least of size 9, while $\overline{9}$ cannot be a part since if this is the case, we have parts $\overline{9}+\overline{5}$, violating the second condition. Now removing parts $\overline{5}$ and 1, subtracting 8 from each of the remaining parts, and preserving all overlines, we again get a partition in $\mathscr{A}_{\{\overline{1}\}}$. Consequently,

$$\begin{aligned}\sum_{\lambda\in\mathscr{A}^{\veebar}_{\{\overline{1}\}}} x^{\sharp(\lambda)}y_1^{\sharp_{2,4}(\lambda)}y_2^{\sharp_{0,4}(\lambda)}z^{\mathsf{O}(\lambda)}q^{|\lambda|} &= \sum_{\lambda\in\mathscr{A}_{\{\overline{1}\}}} x^{\sharp(\lambda)}y_1^{\sharp_{2,4}(\lambda)}y_2^{\sharp_{0,4}(\lambda)}z^{\mathsf{O}(\lambda)}q^{|\lambda|}\\ &\quad + x^2zq^6\sum_{\lambda\in\mathscr{A}_{\{\overline{1}\}}}(xq^8)^{\sharp(\lambda)}y_1^{\sharp_{2,4}(\lambda)}y_2^{\sharp_{0,4}(\lambda)}z^{\mathsf{O}(\lambda)}q^{|\lambda|}.\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{\ell,m,n\ge 0} A(n,m,\ell)x^my^\ell q^n\\ &\quad=\sum_{\lambda\in\mathscr{A}^{\veebar}_{\{\overline{1}\}}} x^{\sharp(\lambda)}(x^{-1}y)^{\sharp_{2,4}(\lambda)}x^{\sharp_{0,4}(\lambda)}y^{\mathsf{O}(\lambda)}q^{|\lambda|}\\ &\quad=\sum_{n_1,n_2,n_3,n_4\ge 0}\frac{x^{n_1+n_2+2n_4}y^{n_2+n_3}q^{n_1+3n_2+2n_3+4n_4}\big(1+x^2yq^{6+8(n_1+n_2+n_3+n_4)}\big)}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^4;q^4)_{n_3}(q^4;q^4)_{n_4}}\\ &\qquad\times q^{4\binom{n_1}{2}+6\binom{n_2}{2}+4\binom{n_3}{2}+8\binom{n_4}{2}+4n_1n_2+4n_1n_3+4n_1n_4+4n_2n_3+4n_2n_4+4n_3n_4}\\ &\quad=(-xq;q^2)_\infty(-yq^2;q^4)_\infty\\ &\quad=\sum_{\ell,m,n\ge 0}B(n,m,\ell)x^my^\ell q^n.\end{aligned}$$

7. Conclusion

The results in this paper together with those in [8–12] make clear that the power of linked partition ideals, first defined in [4], is only now coming into prominence. In addition, the study of linked partition ideals began with an effort to expand the world of partition identities via q-difference equations. This latter topic, considered extensively in [1] and utilized effectively in this paper, should further develop in parallel with the theory of linked partition ideals.

Finally, we see in this paper a new level of partition identity refinement building on the refinements in [7] which in turn refined Glaisher's ancient theorem [14]. It is natural to ask which of the classical partition identities are amenable to refinements and what are the limits of this exploration. We note, for example, that the Rogers–Ramanujan identities themselves have no known refinements along the lines considered here.

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References

1. G. E. Andrews, On q-difference equations for certain well-poised basic hypergeometric series, *Quart. J. Math. Oxford Ser. (2)* **19** (1968), 433–447.
2. G. E. Andrews, Partition identities, *Advances in Math.* **9** (1972), 10–51.
3. G. E. Andrews, An analytic generalization of the Rogers–Ramanujan identities for odd moduli, *Proc. Nat. Acad. Sci. U.S.A.* **71** (1974), 4082–4085.
4. G. E. Andrews, A general theory of identities of the Rogers–Ramanujan type, *Bull. Amer. Math. Soc.* **80** (1974), 1033–1052.
5. G. E. Andrews, Problems and prospects for basic hypergeometric functions, in: *Theory and Application of Special Functions (Proc. Advanced Sem., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1975)*, 191–224, Math. Res. Center, Univ. Wisconsin, Publ. No. **35**, Academic Press, New York, 1975.
6. G. E. Andrews, *The Theory of Partitions*, Reprint of the 1976 original. Cambridge University Press, Cambridge, 1998.
7. G. E. Andrews, Partition identities for k-regular partitions with distinct parts, submitted.
8. G. E. Andrews, S. Chern, and Z. Li, Linked partition ideals and the Alladi–Schur theorem, *J. Combin. Theory Ser. A* **189** (2022), Paper No. 105614, 19 pp.
9. S. Chern, Linked partition ideals, directed graphs and q-multi-summations, *Electron. J. Combin.* **27** (2020), no. 3, Paper No. 3.33, 29 pp.
10. S. Chern, Linked partition ideals and Andrews–Gordon type series for Alladi and Gordon's extension of Schur's identity, *Rocky Mountain J. Math.* **52** (2022), no. 6, 2009–2026.
11. S. Chern, Linked partition ideals and a Schur-type identity of Andrews, in: *Combinatorial and Additive Number Theory V*, 107–117, Springer Proc. Math. Stat., **395**, Springer, Cham, 2022.
12. S. Chern and Z. Li, Linked partition ideals and Kanade–Russell conjectures, *Discrete Math.* **343** (2020), no. 7, Paper No. 111876, 24 pp.
13. S. Corteel and J. Lovejoy, Overpartitions, *Trans. Amer. Math. Soc.* **356** (2004), no. 4, 1623–1635.
14. J. W. L. Glaisher, A theorem in partitions, *Messenger of Math.* **12** (1883), 158–170.
15. B. Gordon, A combinatorial generalization of the Rogers–Ramanujan identities, *Amer. J. Math.* **83** (1961), 393–399.
16. G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Addison-Wesley Publishing Co., Reading, Mass., 1981.
17. S. Ramanujan, Problem 584, *J. Indian Math. Soc.* **6** (1914), 199–200.

18. S. Ramanujan, Proof of certain identities in combinatory analysis, *Proc. Cambridge Philos. Soc.* **19** (1919), 214–216.
19. J.-P. Ramis, About the growth of entire functions solutions of linear algebraic q-difference equations, *Ann. Fac. Sci. Toulouse Math. (6)* **1** (1992), no. 1, 53–94.
20. L. J. Rogers, Third memoir on the expansion of certain infinite products, *Proc. Lond. Math. Soc.* **26** (1894), 15–32.
21. I. Schur, Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche, *Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Klasse* (1917), 302–321.
22. C. Zhang, Une sommation discrète pour des équations aux q-différences linéaires et à coefficients analytiques: théorie générale et exemples (in French), in: *Differential equations and the Stokes phenomenon*, 309–329, World Sci. Publ., River Edge, NJ, 2002.

(G. E. Andrews) Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

E-mail address: `gea1@psu.edu`

(S. Chern) Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, B3H 4R2, Canada

E-mail address: `chenxiaohang92@gmail.com`