Nearly self-conjugate integer partitions

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Abstract. We investigate integer partitions λ of n that are *nearly self-conjugate* in the sense that there are $n-1$ overlapping cells among the Ferrers diagram of λ and its transpose, by establishing a correspondence, through the method of combinatorial telescoping, to partitions of n in which (i). there exists at least one even part; (ii). any even part is of size 2; (iii). the odd parts are distinct; and (iv). no odd part is of size 1. In particular, this correspondence confirms a conjecture that had been given in the OEIS. We also show that the number of the aforementioned overlapping cells has a close connection with partition ranks.

Keywords. Nearly self-conjugate integer partition, symplectic partition, generating function, combinatorial telescoping, partition rank.

2020MSC. 05A17, 05A15.

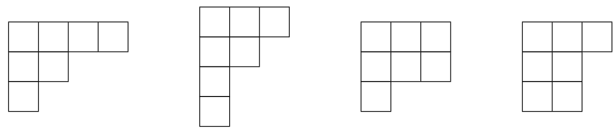
1. Introduction

A *partition* of a natural number n is a nonincreasing sequence of positive integers that sum to n. We say that an integer partition λ is *self-conjugate* if λ is the same as its transpose λ^{T} obtained by reflecting the Ferrers diagram of λ about the main diagonal. Given the elegant simplicity of this definition, it is not surprising that there are many attractive mathematical results concerning the family of self-conjugate integer partitions. In this regard, there is an intimate connection between these integer partitions and the representation theory of the symmetric group. For example, the number $\mathrm{sc}(n)$ of self-conjugate integer partitions of n is the minimal row sum in the character table of S_n, which corresponds to the one-dimensional alternating representation of S_n, and also $\mathrm{sc}(n)$ is the number of conjugacy classes of S_n that split into two classes under restriction to $A_n \leq S_n$; see [8, A000700]. Since the family of self-conjugate integer partitions possesses such interesting combinatorial and representation-theoretic properties, it is natural to consider variations of the definition of a self-conjugate partition.

Our article is inspired in part by [9], in which the concept of a "nearly self-conjugate" set partition is introduced. The term *almost self-conjugate partition* is used in [4] in reference to integer partitions of the form $(\alpha_1 + 1, \ldots, \alpha_d + 1 \mid \alpha_1, \ldots, \alpha_d)$ in Frobenius notation. Since there are important applications of these latter kinds of integer partitions in representation theory, the exploration of combinatorial properties associated with analogous families of integer partitions could conceivably establish new representation-theoretic connections.

Definition 1.1 (Nearly self-conjugate partitions)**.** An integer partition λ is *nearly self-conjugate* if the Ferrers diagrams of λ and its transpose have exactly $n-1$ cells in common.

Example 1.1. There are four nearly self-conjugate partitions of 7, as illustrated below using Ferrers diagrams.



The object of this paper is to confirm a conjecture that was given in 2016, in the OEIS [8] entry indexed as A246581.

Theorem 1.1 (Conjecture in [8, A246581])**.** *Let* $\operatorname{nsc}(n)$ *count the number of nearly self-conjugate partitions of* n*. Then*

$$\sum_{n\geq 0} \operatorname{nsc}(n)q^n = \frac{2q^2}{1-q^2}\prod_{r\geq 1}\left(1+q^{2r+1}\right). \tag{1.1}$$

2. Proof of Theorem 1.1

2.1. Nearly self-conjugate partitions. Let $\mathscr{N}(n)$ denote the set of nearly self-conjugate partitions of n. Let $\mathscr{O}(n)$ denote the set of partitions of n with exactly one even part such that the differences between parts are at least 2.

We begin by making use of an analogue of the famous proof of a classical relation due to Sylvester [1, p. 14, Entry 8], claiming that the number of self-conjugate partitions of n equals the number of partitions of n into distinct odd parts.

Lemma 2.1. *The number of nearly self-conjugate partitions of* n *is equal to twice the number of partitions of* n *with exactly one even part such that the differences between parts are at least* 2*.*

Proof. Define the equivalence relation $\sim$ on $\mathscr{N}(n)$ so that for partitions μ and ν in $\mathscr{N}(n)$, $\mu \sim \nu$ if and only if μ and ν are equal or are transposes of one another. Notice that in each equivalence class, there are exactly two partitions in $\mathscr{N}(n)$.

Given a partition $\lambda \in \mathscr{O}(n)$, let $\phi_n(\lambda)$ denote the partition obtained by:

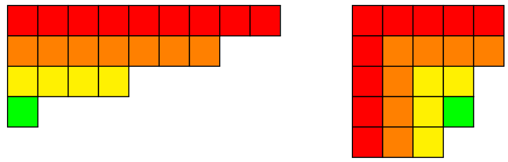
(i). Folding any odd part of size $2k-1$ in λ (which is a stripe of $2k-1$ cells) as a hook shape so that there are $k-1$ cells below the corner of the hook and $k-1$ cells to the right of the corner;

(ii). Folding any even part of size $2k$ in λ (which is a stripe of $2k$ cells) as a hook shape so that there are k cells below the corner of the hook and $k-1$ cells to the right of the corner;

(iii). Nesting these hook shapes by aligning their corners.

It is easily seen that the mapping ϕ_n induces a well-defined bijection from $\mathscr{O}(n)$ to the quotient $\mathscr{N}(n)/\sim$. □

Example 2.1. We see that the tuple $(9,7,4,1)$ is an integer partition of 23 with exactly one even part such that the differences between parts are at least 2. The application of the above procedure to this partition in $\mathscr{O}(23)$ is illustrated below.



2.2. Symplectic partitions. To make use of Lemma 2.1 for a proof of Theorem 1.1, we shift our attention to the right hand side of (1.1).

Definition 2.1 (Symplectic partitions)**.** A *symplectic partition* is an integer partition λ such that

(i). There exists at least one even part in λ;
(ii). Any even part in λ is of size 2,;
(iii). The odd parts in λ are distinct;
(iv). No odd part in λ is of size 1.

Example 2.2. There are three symplectic partitions of 9. Namely, $(7,2)$, $(5,2,2)$ and $(3,2,2,2)$.

Let $\mathrm{Sp}(n)$ count the number of symplectic partitions of n. Then

$$\sum_{n\geq 0}\mathrm{Sp}(n)q^n=\frac{q^2}{1-q^2}\prod_{r\geq 1}\left(1+q^{2r+1}\right), \tag{2.1}$$

which is exactly half the right hand side of (1.1).

Remark. The name of symplectic partitions is borrowed from a result of Rudvalis and Shinoda [7]. Let $Sp=Sp(n)$ be the classical symplectic group acting on an n-dimensional vector space V over a finite field F_p in its natural way. Let $P_{Sp,n}(2,p)$ be the chance that an element of Sp fixes a two-dimensional subspace and let $P_{Sp,\infty}(2,p)$ be the $n\to\infty$ limit of $P_{Sp,n}(2,p)$. Then, with $q=\frac{1}{p}$,

$$P_{Sp,\infty}(2,p)=\frac{q^3}{(1-q)(1-q^2)}\prod_{r\geq 1}\frac{1}{1+q^r}.$$

See [6, p. 70, eq. (3) with $k=2$]. Notice also that as a formal power series in q,

$$\sum_{n\geq 0}\mathrm{Sp}(n)(-q)^n=\frac{q^2}{1-q^2}\prod_{r\geq 1}\left(1-q^{2r+1}\right)=\frac{q^2}{(1-q)(1-q^2)}\prod_{r\geq 1}\frac{1}{1+q^r}.$$

One may compare their resemblance.

Our object here is the following relation.

Lemma 2.2. *The number of partitions of n with exactly one even part such that the differences between parts are at least 2 is equal to the number of symplectic partitions of n.*

Proof. Let $\mathscr{S}(n)$ denote the set of symplectic partitions of n. Recall also that $\mathscr{O}(n)$ is the set of partitions of n with exactly one even part such that the differences between parts are at least 2. We are going to establish a correspondence between $\mathscr{S}(n)$ and $\mathscr{O}(n)$.

First, for each positive odd integer $2m-1$, we denote by $\mathscr{A}_{2m-1}(n)$ the set of partitions λ of n such that

(i). Any even part in λ is of size 2;
(ii). The (uncolored) odd parts in λ are distinct;
(iii). There is always an extra colored part of size 1, denoted by 1_c; and
(iv). No part in λ is of size $2m-1$.

For example, there are three partitions in $\mathscr{A}_3(9)$. Namely, $(7,1,1_c)$, $(5,2,1,1_c)$ and $(2,2,2,2,1_c)$.

We also denote by $\mathscr{O}_{2m}(n)$ the subset of partitions in $\mathscr{O}(n)$ with the only even part of size $2m$.

Our next objective is to establish a correspondence between $\mathscr{A}_{2m+1}(n)$ and $\mathscr{A}_{2m-1}(n) \cup \mathscr{O}_{2m}(n)$ for each $m \geq 1$. We observe that the partitions λ of n satisfying the above conditions (i)–(iii) and the new condition

(iv$^\diamondsuit$). No part in λ is of size $2m-1$ and $2m+1$,

are in both $\mathscr{A}_{2m+1}(n)$ and $\mathscr{A}_{2m-1}(n)$. Let $\mathscr{A}^\diamondsuit_{2m-1}(n)$ be the set of such partitions of n. We subtract $\mathscr{A}^\diamondsuit_{2m-1}(n)$ from the two partition sets $\mathscr{A}_{2m+1}(n)$ and $\mathscr{A}_{2m-1}(n)$, respectively. For what are left, we denote by $\mathscr{A}^\vartriangle_{2m-1}(n)$ the set of partitions λ of n satisfying the conditions (i)–(iii) and

(iv$^\vartriangle$). $2m+1$ is not a part in λ and $2m-1$ is a part in λ.

Also, we denote by $\mathscr{A}^\triangledown_{2m-1}(n)$ the set of partitions λ of n satisfying the conditions (i)–(iii) and

(iv$^\triangledown$). $2m-1$ is not a part in λ and $2m+1$ is a part in λ.

Now, it suffices to establish a correspondence between $\mathscr{A}^\vartriangle_{2m-1}(n)$ and $\mathscr{A}^\triangledown_{2m-1}(n) \cup \mathscr{O}_{2m}(n)$. For any partition in $\mathscr{A}^\triangledown_{2m-1}(n)$, we have a part of size $(2m+1)$, and therefore we may split it into a part of size $(2m-1)$ and a part of size 2, thereby yielding a partition in $\mathscr{A}^\vartriangle_{2m-1}(n)$ with 2 appearing at least once.

Thus, it remains to establish a correspondence between partitions in $\mathscr{O}_{2m}(n)$ and partitions in $\mathscr{A}^\vartriangle_{2m-1}(n)$ with no even parts. For any partition of the latter form, it has no part of size $(2m+1)$, one part of size $(2m-1)$ and one colored part 1_c. Combining the parts $(2m-1)$ and 1_c then gives a part of size $2m$. For this resulting partition, there are no parts of sizes $(2m \pm 1)$. Thus, it is in $\mathscr{O}_{2m}(n)$.

Since we have a correspondence between $\mathscr{A}_{2m+1}(n)$ and $\mathscr{A}_{2m-1}(n) \cup \mathscr{O}_{2m}(n)$ for each $m \geq 1$, we may bijectively map $\sqcup_{m\geq 1}\mathscr{A}_{2m+1}(n)$ to $(\sqcup_{m\geq 1}\mathscr{A}_{2m-1}(n)) \cup (\sqcup_{m\geq 1}\mathscr{O}_{2m}(n))$. Subtracting $\sqcup_{m\geq 1}\mathscr{A}_{2m-1}(n)$ from $\sqcup_{m\geq 1}\mathscr{A}_{2m+1}(n)$, we are left with partitions of n satisfying the conditions (i)–(iii) in this proof with an extra constraint that an uncolored 1 must appear as a part and it appears exactly once. But for such partitions, we also have a colored part 1_c. Combining the parts 1 and 1_c gives a part of size 2. In other words, we arrive at partitions of n such that any even part is of size 2 (appearing at least once by the condition (i) and the construction of $2 = 1 + 1_c$), the odd parts are distinct (by the condition (ii)) and 1 does not appear as a part (by the fact that 1 and 1_c are absorbed in the construction of $2 = 1 + 1_c$). Such partitions are exactly in $\mathscr{S}(n)$.

Hence, we have a correspondence between $\mathscr{O}(n) = \sqcup_{m\geq 1}\mathscr{O}_{2m}(n)$ and $\mathscr{S}(n)$. Our lemma is therefore established. □

Remark. The above proof can be understood as an instance of the method of combinatorial telescoping [3].

2.3. Proof of Theorem 1.1. By Lemmas 2.1 and 2.2, we have

$$\mathrm{nsc}(n) = 2\,\mathrm{Sp}(n).$$

Theorem 1.1 follows by recalling (2.1).

We also want to point out that the proof of Lemma 2.2 establishes combinatorially the following identity:

$$\left(\prod_{r\geq 1}\left(1+q^{2r-1}\right)\right)\sum_{m\geq 1}\frac{q^{2m}}{(1+q^{2m-1})(1+q^{2m+1})} = \frac{q^2}{1-q^2}\prod_{r\geq 1}\left(1+q^{2r+1}\right).$$

This can also be shown analytically as follows.

$$\begin{aligned}
&\left(\prod_{r\geq 1}\left(1+q^{2r-1}\right)\right)\sum_{m\geq 1}\frac{q^{2m}}{(1+q^{2m-1})(1+q^{2m+1})}\\
&\quad= \left(\prod_{r\geq 1}\left(1+q^{2r-1}\right)\right)\cdot\frac{1}{q-q^{-1}}\sum_{m\geq 1}\left(\frac{1}{1+q^{2m-1}}-\frac{1}{1+q^{2m+1}}\right)\\
&\quad= \left(\prod_{r\geq 1}\left(1+q^{2r-1}\right)\right)\cdot\frac{1}{q-q^{-1}}\cdot\left(\frac{1}{1+q}-1\right)\\
&\quad= \left(\prod_{r\geq 1}\left(1+q^{2r+1}\right)\right)\cdot\frac{q^2}{1-q^2}.
\end{aligned}$$

3. Overlapping differences

In general, it is natural to investigate partitions of n such that the Ferrers diagrams of the partition and its transpose have exactly $n-k$ cells in common when $k\geq 2$. For these cases, although we seem to have no generating function identities as neat as (2.1), it is still possible to make a connection with other partition statistics.

Given an integer partition λ, we define its *overlapping difference*, denoted by $\mathrm{Diff}(\lambda)$, as the size of λ minus the number of overlapping cells among the Ferrers diagrams of λ and its transpose. For example, the overlapping difference of any self-conjugate partition is 0, and the overlapping difference of any nearly self-conjugate partition is 1.

In [2], Atkin extended the Dyson's partition rank [5] and introduced the *k-th rank* of a partition λ, denoted by $\mathrm{Rank}_k(\lambda)$, as the k-th largest part minus the number of parts greater than or equal to k. Then Dyson's rank is the first rank in Atkin's terminology.

Proposition 3.1. *For any partition λ,*

$$\mathrm{Diff}(\lambda) = \sum_{k=1}^{D(\lambda)} |\,\mathrm{Rank}_k(\lambda)|, \tag{3.1}$$

where $D(\lambda)$ is the size of the Durfee square of λ.

Proof. We make use of the *Frobenius symbol* of a partition, which is a two-rowed array

$$\begin{pmatrix} s_1 & s_2 & \cdots & s_d \\ t_1 & t_2 & \cdots & t_d \end{pmatrix}$$

with $s_1 > s_2 > \cdots > s_d \geq 0$ and $t_1 > t_2 > \cdots > t_d \geq 0$, where s_k (resp. t_k) counts the number of cells to the right of (resp. below) the k-th diagonal entry of the Durfee square of the partition in its Ferrers diagram.

Assume that a partition λ has the above Frobenius symbol representation. Then

$$|\lambda| = \sum_{k=1}^{d} \big(1 + s_k + t_k\big).$$

Also, $D(\lambda) = d$, and for each $1 \leq k \leq d$,

$$\operatorname{Rank}_k(\lambda) = s_k - t_k.$$

We next observe that the number of overlapping cells on the k-th diagonal hooks of λ and its transpose equals $1 + 2\min(s_k, t_k)$. Hence,

$$\begin{aligned} \operatorname{Diff}(\lambda) &= |\lambda| - \sum_{k=1}^{d} \big(1 + 2\min(s_k, t_k)\big) \\ &= \sum_{k=1}^{d} \big(1 + s_k + t_k\big) - \sum_{1 \leq k \leq d} \big(1 + s_k + t_k - |s_k - t_k|\big) \\ &= \sum_{k=1}^{d} |s_k - t_k| \\ &= \sum_{k=1}^{d} |\operatorname{Rank}_k(\lambda)|. \end{aligned}$$

This establishes our desired result. □

4. Conclusion

We conclude with a combinatorial problem given as follows. Recall that the number $\operatorname{sc}(n)$ of self-conjugate integer partitions of n has the generating function

$$\sum_{n \geq 0} \operatorname{sc}(n) q^n = \prod_{r \geq 0} \left(1 + q^{2r+1}\right).$$

Therefore, we observe from Theorem 1.1 that

$$\sum_{n \geq 0} \operatorname{nsc}(n) q^n = \frac{2q^2}{(1+q)(1-q^2)} \sum_{n \geq 0} \operatorname{sc}(n) q^n.$$

Noticing that

$$\frac{q^2}{(1+q)(1-q^2)} = \sum_{n \geq 0} (-1)^n \left\lfloor \frac{n}{2} \right\rfloor q^n,$$

we have the following relation concerning $\operatorname{nsc}(n)$ and $\operatorname{sc}(n)$.

Proposition 4.1. *For $n \geq 0$,*

$$\frac{1}{2}\,\mathrm{nsc}(n) = \sum_{k=0}^{n} (-1)^{n-k} \left\lfloor \frac{n-k}{2} \right\rfloor \mathrm{sc}(k). \tag{4.1}$$

A combinatorial proof of this relation would be appealing.

Acknowledgements. The authors want to thank George Beck for many useful comments that he had provided concerning this article. The second author was supported by a Killam Postdoctoral Fellowship from the Killam Trusts.

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