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Weighted partition rank and crank moments. I. Andrews–Beck type congruences

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Dedicated to Professor Bruce Berndt on the occasion of his 80th birthday.

Abstract. Recently, George Beck conjectured and George Andrews proved a handful of novel congruences concerning the rank statistic of Dyson and the crank statistic of Andrews and Garvan. In this paper, we shall prove two identities concerning the weighted rank and crank moments, from which more congruences of Andrews–Beck type may be deduced.

Keywords. Partition, Andrews–Beck type congruence, rank, crank, weighted moment.

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1. Introduction

1.1. Background. As usual, a *partition* of a positive integer n is a weakly decreasing sequence of positive integers whose sum equals n . For example, 4 has five partitions: 4, 3 + 1, 2 + 2, 2 + 1 + 1 and 1 + 1 + 1 + 1. If we denote by $p(n)$ the number of partitions of n , then $p(4) = 5$.

In the theory of partitions, one of the most fascinating results is due to Ramanujan, who discovered that $p(n)$ satisfies the following congruences:

$$p(5n + 4) \equiv 0 \pmod{5}, \tag{1.1}$$

$$p(7n + 5) \equiv 0 \pmod{7}, \tag{1.2}$$

$$p(11n + 6) \equiv 0 \pmod{11}. \tag{1.3}$$

See [13–15] or [7] or [1, Chapter 10].

In his famous 1944 paper [8], Dyson defined the *rank* of a partition as the largest part minus the number of parts. Dyson then conjectured that the rank statistic may provide combinatorial interpretations of (1.1) and (1.2); this assertion was later confirmed by Atkin and Swinnerton-Dyer [6]. Dyson also conjectured the existence of a *crank* statistic, which is able to combinatorially explain all the three congruences of Ramanujan. Such a statistic was not discovered until over four decades later by Andrews and Garvan [4] based on Garvan’s study of the vector crank [10, 11].

As in [6], we denote by $N(m, k, n)$ the number of partitions of n with rank congruent to m modulo k . Atkin and Swinnerton-Dyer proved (1.1) and (1.2) by showing that for $0 \leq i \leq 4$,

$$N(i, 5, 5n + 4) = \frac{1}{5}p(5n + 4)$$

and that for $0 \leq i \leq 6$,

$$N(i, 7, 7n + 5) = \frac{1}{7}p(7n + 5).$$

In a recent paper [3], Andrews recorded that George Beck has conjectured a number of new congruences along a somewhat different road. Instead of considering the $N(m, k, n)$ function, Beck studied the total number of parts in the partitions of n with rank congruent to m modulo k , which is defined by $NT(m, k, n)$. One of the results proved by Andrews reads as follows.

Theorem (Andrews [3, Theorem 1.1]). *If $i = 1$ or 4 , then for $n \geq 0$,*

$$\begin{aligned} & NT(1, 5, 5n + i) + 2NT(2, 5, 5n + i) \\ & - 2NT(3, 5, 5n + i) - NT(4, 5, 5n + i) \equiv 0 \pmod{5}. \end{aligned} \quad (1.4)$$

Andrews also remarked that (1.4) is trivial if one replaces the $NT(m, k, n)$ function by the rank function $N(m, k, n)$. This is simply due to the symmetry

$$N(m, k, n) = N(k - m, k, n).$$

However, the above symmetry is generally false for the $NT(m, k, n)$ function. Therefore, the validness of (1.4) is in some sense exciting.

In his proof of (1.4), Andrews did not apply the differentiation technique directly to the trivariate generating function

$$\sum_{n \geq 0} \sum_{\lambda \vdash n} x^{\#\lambda} z^{\text{rank}(\lambda)} q^n = \sum_{n \geq 0} \frac{x^n q^{n^2}}{(zq; q)_n (xq/z; q)_n}.$$

Instead, he transformed the above generating function as

$$\begin{aligned} \sum_{n \geq 0} \sum_{\lambda \vdash n} x^{\#\lambda} z^{\text{rank}(\lambda)} q^n &= 1 + \frac{1}{(xq; q)_\infty} \sum_{n \geq 1} (-1)^{n-1} q^{n(3n+1)/2} x^n \frac{(xq; q)_n}{(q; q)_{n-1}} \\ &\quad \times \left(\frac{1}{q^n(1 - zq^n)} + \frac{x}{z(1 - \frac{xq^n}{z})} \right), \end{aligned}$$

so he may take advantage of several results proved already by Atkin and Swinnerton-Dyer [6]; see [3, Theorem 3.1].

However, we discover that if we consider a weighted rank moment, then it is possible to arrive at a connection with the second Atkin–Garvan rank moment defined in [5] so that, surprisingly, (1.4) follows as an immediate consequence. In analogy, a weighted crank moment will lead to proofs of several conjectures of Beck concerning the crank function.

1.2. Notation and terminology. Let \mathcal{P} be the set of partitions. As usual, the notation $\lambda \vdash n$ means that λ is a partition of n . Below, let λ always be a partition. Let $|\lambda|$ be the size of λ . We use $\#\lambda$ and $\omega(\lambda)$ to denote the number of parts in λ and the number of ones in λ , respectively. Further, $\text{rank}(\lambda)$ and $\text{crank}(\lambda)$ denote the rank and crank of λ .

As already defined, $NT(m, k, n)$ equals the total number of parts in the partitions of n with rank congruent to m modulo k . We also denote by $N(m, n)$ the number

of partitions of n whose rank is m . Then the second Atkin–Garvan rank moment $N_2(n)$ is defined by

$$N_2(n) := \sum_{m=-\infty}^{\infty} m^2 N(m, n) = \sum_{\lambda \vdash n} \text{rank}^2(\lambda).$$

On the other hand, let $M_\omega(m, k, n)$ count the total number of ones in the partitions of n with crank congruent to m modulo k .

Finally, we adopt the standard q -Pochhammer symbol for $n \in \mathbb{N} \cup \{\infty\}$:

$$(A; q)_n := \prod_{k=0}^{n-1} (1 - Aq^k).$$

1.3. Main results. Our first result treats the following weighted rank moment.

Theorem 1.1. *We have*

$$\sum_{\lambda \in \mathcal{P}} \#(\lambda) \text{rank}(\lambda) q^{|\lambda|} = - \sum_{n \geq 1} \frac{q^{n^2}}{(q; q)_n^2} \sum_{m=1}^n \frac{q^m}{(1 - q^m)^2} \quad (1.5)$$

and

$$\sum_{\lambda \vdash n} \#(\lambda) \text{rank}(\lambda) = -\frac{1}{2} N_2(n). \quad (1.6)$$

Remark 1.1. It is worth pointing out that the following generating function identity for $N_2(n)$ is used most frequently:

$$\sum_{n \geq 0} N_2(n) q^n = -\frac{2}{(q; q)_\infty} \sum_{n \geq 1} \frac{(-1)^n q^{n(3n+1)/2} (1 + q^n)}{(1 - q^n)^2}.$$

See [2, Eq. (3.4)].

Likewise, for the weighted crank moment, our result reads as follows.

Theorem 1.2. *We have*

$$\sum_{\lambda \in \mathcal{P}} \omega(\lambda) \text{crank}(\lambda) q^{|\lambda|} = -\frac{1}{(q; q)_\infty} \sum_{n \geq 1} \frac{q^n}{(1 - q^n)^2} \quad (1.7)$$

and

$$\sum_{\lambda \vdash n} \omega(\lambda) \text{crank}(\lambda) = -np(n). \quad (1.8)$$

Remark 1.2. Let $M(m, n)$ denote number of partitions of n with crank m . Atkin and Garvan [5] defined the k -th crank moment $M_k(n)$ by

$$M_k(n) = \sum_{m=-\infty}^{\infty} m^k M(m, n) = \sum_{\lambda \vdash n} \text{crank}^k(\lambda).$$

It can be shown by means of a relation due to Dyson [9] that

$$M_2(n) = 2np(n).$$

It turns out that

$$\sum_{\lambda \vdash n} \omega(\lambda) \text{crank}(\lambda) = -\frac{1}{2} M_2(n). \quad (1.9)$$

This is an analog of (1.6).

As consequences of Theorems 1.1 and 1.2, we arrive at a number of Andrews–Beck type congruences with (1.4) included.

Corollary 1.3. *If $i = 1$ or 4 , then for $n \geq 0$,*

$$\begin{aligned} & NT(1, 5, 5n + i) + 2NT(2, 5, 5n + i) \\ & - 2NT(3, 5, 5n + i) - NT(4, 5, 5n + i) \equiv 0 \pmod{5}. \end{aligned} \quad (1.10)$$

Corollary 1.4. *If $i = 1$ or 5 , then for $n \geq 0$,*

$$\begin{aligned} & NT(1, 7, 7n + i) + 2NT(2, 7, 7n + i) \\ & + 3NT(3, 7, 7n + i) - 3NT(4, 7, 7n + i) \\ & - 2NT(5, 7, 7n + i) - NT(6, 7, 7n + i) \equiv 0 \pmod{7}. \end{aligned} \quad (1.11)$$

Corollary 1.5. *If $i = 0$ or 4 , then for $n \geq 0$,*

$$\begin{aligned} & M_\omega(1, 5, 5n + i) + 2M_\omega(2, 5, 5n + i) \\ & - 2M_\omega(3, 5, 5n + i) - M_\omega(4, 5, 5n + i) \equiv 0 \pmod{5}. \end{aligned} \quad (1.12)$$

Corollary 1.6. *If $i = 0$ or 5 , then for $n \geq 0$,*

$$\begin{aligned} & M_\omega(1, 7, 7n + i) + 2M_\omega(2, 7, 7n + i) \\ & + 3M_\omega(3, 7, 7n + i) - 3M_\omega(4, 7, 7n + i) \\ & - 2M_\omega(5, 7, 7n + i) - M_\omega(6, 7, 7n + i) \equiv 0 \pmod{7}. \end{aligned} \quad (1.13)$$

Corollary 1.7. *If $i = 0$ or 6 , then for $n \geq 0$,*

$$\begin{aligned} & M_\omega(1, 11, 11n + i) + 2M_\omega(2, 11, 11n + i) + 3M_\omega(3, 11, 11n + i) \\ & + 4M_\omega(4, 11, 11n + i) + 5M_\omega(5, 11, 11n + i) - 5M_\omega(6, 11, 11n + i) \\ & - 4M_\omega(7, 11, 11n + i) - 3M_\omega(8, 11, 11n + i) - 2M_\omega(9, 11, 11n + i) \\ & - M_\omega(10, 11, 11n + i) \equiv 0 \pmod{11}. \end{aligned} \quad (1.14)$$

Finally, let $\text{spt}(n)$ denote the total number of appearances of the smallest parts in all partitions of n . It was shown by Andrews [2] that

$$\text{spt}(n) = np(n) - \frac{1}{2}N_2(n).$$

In view of (1.6) and (1.8), we have the following intriguing relation.

Corollary 1.8. *For $n \geq 0$,*

$$\text{spt}(n) = \sum_{\lambda \vdash n} \#(\lambda) \text{rank}(\lambda) - \sum_{\lambda \vdash n} \omega(\lambda) \text{crank}(\lambda). \quad (1.15)$$

2. The weighted rank moment

We first study the weighted rank moment. It was shown in [3] that

$$\sum_{n \geq 0} \sum_{\lambda \vdash n} x^{\#(\lambda)} z^{\text{rank}(\lambda)} q^n = \sum_{n \geq 0} \frac{x^n q^{n^2}}{(zq; q)_n (xq/z; q)_n}. \quad (2.1)$$

Proof of Theorem 1.1. We first apply the operator $[\partial/\partial x]_{x=1}$ to (2.1).

$$\begin{aligned}
& \sum_{n \geq 0} \sum_{\lambda \vdash n} \#(\lambda) z^{\text{rank}(\lambda)} q^n \\
&= \sum_{n \geq 0} \left[\frac{\partial}{\partial x} \frac{x^n q^{n^2}}{(zq; q)_n (xq/z; q)_n} \right]_{x=1} \\
&= \sum_{n \geq 0} \left[\frac{x^n q^{n^2}}{(zq; q)_n (xq/z; q)_n} \frac{\partial}{\partial x} \log \left(\frac{x^n}{(xq/z; q)_n} \right) \right]_{x=1} \\
&= \sum_{n \geq 0} \frac{q^{n^2}}{(zq; q)_n (q/z; q)_n} \left[\frac{\partial}{\partial x} \left(n \log x - \sum_{m=1}^n \log(1 - xq^m/z) \right) \right]_{x=1} \\
&= \sum_{n \geq 0} \frac{q^{n^2}}{(zq; q)_n (q/z; q)_n} \left[\frac{n}{x} + \sum_{m=1}^n \frac{q^m}{z - xq^m} \right]_{x=1} \\
&= \sum_{n \geq 1} \frac{q^{n^2}}{(zq; q)_n (q/z; q)_n} \left(n + \sum_{m=1}^n \frac{q^m}{z - q^m} \right). \tag{2.2}
\end{aligned}$$

Next, we make an easy observation: for any $n \in \mathbb{N} \cup \{\infty\}$,

$$\left[\frac{\partial}{\partial z} \log \left(\frac{1}{(zq; q)_n (q/z; q)_n} \right) \right]_{z=1} = \left[\sum_{m=1}^n \left(\frac{q^m}{1 - zq^m} + \frac{q^m}{zq^m - z^2} \right) \right]_{z=1} = 0. \tag{2.3}$$

Applying the operator $[\partial/\partial z]_{z=1}$ to (2.2) and making use of (2.3), we have

$$\begin{aligned}
& \sum_{n \geq 0} \sum_{\lambda \vdash n} \#(\lambda) \text{rank}(\lambda) q^n \\
&= \sum_{n \geq 1} \left[\frac{\partial}{\partial z} \frac{q^{n^2}}{(zq; q)_n (q/z; q)_n} \left(n + \sum_{m=1}^n \frac{q^m}{z - q^m} \right) \right]_{z=1} \\
&= \sum_{n \geq 1} \left[\frac{nq^{n^2}}{(zq; q)_n (q/z; q)_n} \frac{\partial}{\partial z} \log \left(\frac{1}{(zq; q)_n (q/z; q)_n} \right) \right]_{z=1} \\
&\quad + \sum_{n \geq 1} \sum_{m=1}^n \left[\frac{q^{n^2}}{(zq; q)_n (q/z; q)_n} \frac{q^m}{z - q^m} \frac{\partial}{\partial z} \log \left(\frac{1}{(zq; q)_n (q/z; q)_n (z - q^m)} \right) \right]_{z=1} \\
&= - \sum_{n \geq 1} \frac{q^{n^2}}{(q; q)_n^2} \sum_{m=1}^n \frac{q^m}{(1 - q^m)^2}. \tag{2.4}
\end{aligned}$$

This is the first part of Theorem 1.1.

On the other hand, if one applies the operator $[\frac{\partial}{\partial z} (z \frac{\partial}{\partial z})]_{z=1}$ to

$$\sum_{n \geq 0} \sum_{\lambda \vdash n} z^{\text{rank}(\lambda)} q^n = \sum_{n \geq 0} \frac{q^{n^2}}{(zq; q)_n (q/z; q)_n},$$

one shall find that

$$\sum_{n \geq 0} N_2(n) q^n = \sum_{n \geq 0} \sum_{\lambda \vdash n} \text{rank}(\lambda)^2 q^n$$

$$\begin{aligned}
&= \sum_{n \geq 0} \left[\frac{\partial}{\partial z} \left(z \frac{\partial}{\partial z} \frac{q^{n^2}}{(zq; q)_n (q/z; q)_n} \right) \right]_{z=1} \\
&= \sum_{n \geq 0} \left[\frac{\partial}{\partial z} \frac{zq^{n^2}}{(zq; q)_n (q/z; q)_n} \sum_{m=1}^n \left(\frac{q^m}{1-zq^m} + \frac{q^m}{zq^m - z^2} \right) \right]_{z=1} \\
&= 2 \sum_{n \geq 1} \frac{q^{n^2}}{(q; q)_n^2} \sum_{m=1}^n \frac{q^m}{(1-q^m)^2}. \tag{2.5}
\end{aligned}$$

This combining with (2.4) gives the second part of Theorem 1.1. \square

The arithmetic properties satisfied by $N_2(n)$ have been well studied. For example, it was indicated in [12, p. 285] that

$$N_2(5n+1) \equiv N_2(5n+4) \equiv 0 \pmod{5}$$

and

$$N_2(7n+1) \equiv N_2(7n+5) \equiv 0 \pmod{7}.$$

On the other hand, it is trivial to see that

$$\begin{aligned}
\sum_{\lambda \in \mathcal{P}} \#(\lambda) \text{rank}(\lambda) q^{|\lambda|} &\equiv \sum_{n \geq 0} (NT(1, 5, n) + 2NT(2, 5, n) \\
&\quad - 2NT(3, 5, n) - NT(4, 5, n)) q^n \pmod{5}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\lambda \in \mathcal{P}} \#(\lambda) \text{rank}(\lambda) q^{|\lambda|} &\equiv \sum_{n \geq 0} (NT(1, 7, n) + 2NT(2, 7, n) \\
&\quad + 3NT(3, 7, n) - 3NT(4, 7, n) \\
&\quad - 2NT(5, 7, n) - NT(6, 7, n)) q^n \pmod{7}.
\end{aligned}$$

Therefore, Corollaries 1.3 and 1.4 follow from the above.

3. The weighted crank moment

Recall that the crank of a partition λ is defined by

$$\text{crank}(\lambda) = \begin{cases} \ell(\lambda) & \text{if } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0, \end{cases}$$

where $\omega(\lambda)$ denotes the number of ones in λ as before, $\ell(\lambda)$ denotes the largest part in λ and $\mu(\lambda)$ denotes the number of parts in λ that are larger than $\omega(\lambda)$.

As in [4], we have

$$\begin{aligned}
\sum_{n \geq 0} \sum_{\lambda \vdash n} x^{\omega(\lambda)} z^{\text{crank}(\lambda)} q^n &= \frac{1-q}{(zq; q)_\infty} + \sum_{j \geq 1} \frac{x^j q^j z^{-j}}{(q^2; q)_{j-1} (zq^{j+1}; q)_\infty} \\
&= \frac{1-q}{(zq; q)_\infty} \sum_{j \geq 0} \frac{(zq; q)_j}{(q; q)_j} \left(\frac{xq}{z} \right)^j \\
&= \frac{(1-q)(xq^2; q)_\infty}{(zq; q)_\infty (xq/z; q)_\infty}. \tag{3.1}
\end{aligned}$$

Here in the last equality we make use of the q -binomial theorem [1, Theorem 2.1]:

$$\sum_{n \geq 0} \frac{(a; q)_n t^n}{(q; q)_n} = \frac{(at; q)_\infty}{(t; q)_\infty}.$$

If we take $x = 1$ in (3.1), we recover the bivariate generating function in [4].

Proof of Theorem 1.2. This time we apply the operator $[\partial/\partial x]_{x=1}$ to (3.1).

$$\begin{aligned} & \sum_{n \geq 0} \sum_{\lambda \vdash n} \omega(\lambda) z^{\text{crank}(\lambda)} q^n \\ &= \left[\frac{\partial}{\partial x} \frac{(1-q)(xq^2; q)_\infty}{(zq; q)_\infty (xq/z; q)_\infty} \right]_{x=1} \\ &= \left[\frac{(1-q)(xq^2; q)_\infty}{(zq; q)_\infty (xq/z; q)_\infty} \frac{\partial}{\partial x} \log \left(\frac{(xq^2; q)_\infty}{(xq/z; q)_\infty} \right) \right]_{x=1} \\ &= \frac{(q; q)_\infty}{(zq; q)_\infty (q/z; q)_\infty} \left[\frac{\partial}{\partial x} \sum_{n \geq 1} (\log(1 - xq^{n+1}) - \log(1 - xq^n/z)) \right]_{x=1} \\ &= \frac{(q; q)_\infty}{(zq; q)_\infty (q/z; q)_\infty} \sum_{n \geq 1} \left(-\frac{q^{n+1}}{1 - q^{n+1}} + \frac{q^n/z}{1 - q^n/z} \right). \end{aligned} \quad (3.2)$$

We then apply the operator $[\partial/\partial z]_{z=1}$ to (3.2) and use (2.3) to deduce

$$\begin{aligned} & \sum_{n \geq 0} \sum_{\lambda \vdash n} \omega(\lambda) \text{crank}(\lambda) q^n \\ &= \left[\frac{\partial}{\partial z} \frac{(q; q)_\infty}{(zq; q)_\infty (q/z; q)_\infty} \sum_{n \geq 1} \left(-\frac{q^{n+1}}{1 - q^{n+1}} + \frac{q^n/z}{1 - q^n/z} \right) \right]_{z=1} \\ &= \sum_{n \geq 1} \left[\frac{(q; q)_\infty}{(zq; q)_\infty (q/z; q)_\infty} \cdot \frac{q^n/z}{1 - q^n/z} \cdot \frac{1}{q^n - z} \right]_{z=1} \\ &= -\frac{1}{(q; q)_\infty} \sum_{n \geq 1} \frac{q^n}{(1 - q^n)^2}. \end{aligned} \quad (3.3)$$

To prove the second part of Theorem 1.2, we simply observe that

$$\begin{aligned} \sum_{n \geq 0} np(n)q^n &= \left[\frac{\partial}{\partial z} \frac{1}{(zq; zq)_\infty} \right]_{z=1} \\ &= \frac{1}{(q; q)_\infty} \sum_{n \geq 1} \frac{nq^n}{1 - q^n} \\ &= \frac{1}{(q; q)_\infty} \sum_{n \geq 1} \frac{q^n}{(1 - q^n)^2}. \end{aligned}$$

In view of (3.3), we arrive at the desired result. \square

Once again, we notice that

$$\begin{aligned} \sum_{\lambda \in \mathcal{P}} \omega(\lambda) \text{crank}(\lambda) q^{|\lambda|} &\equiv \sum_{n \geq 0} (M_\omega(1, 5, n) + 2M_\omega(2, 5, n) \\ &\quad - 2M_\omega(3, 5, n) - M_\omega(4, 5, n)) q^n \pmod{5}, \end{aligned}$$

$$\begin{aligned} \sum_{\lambda \in \mathcal{P}} \omega(\lambda) \operatorname{crank}(\lambda) q^{|\lambda|} &\equiv \sum_{n \geq 0} (M_\omega(1, 7, n) + 2M_\omega(2, 7, n) \\ &\quad + 3M_\omega(3, 7, n) - 3M_\omega(4, 7, n) \\ &\quad - 2M_\omega(5, 7, n) - M_\omega(6, 7, n)) q^n \pmod{7} \end{aligned}$$

and

$$\begin{aligned} \sum_{\lambda \in \mathcal{P}} \omega(\lambda) \operatorname{crank}(\lambda) q^{|\lambda|} &\equiv \sum_{n \geq 0} (M_\omega(1, 11, n) + 2M_\omega(2, 11, n) + 3M_\omega(3, 11, n) \\ &\quad + 4M_\omega(4, 11, n) + 5M_\omega(5, 11, n) - 5M_\omega(6, 11, n) \\ &\quad - 4M_\omega(7, 11, n) - 3M_\omega(8, 11, n) - 2M_\omega(9, 11, n) \\ &\quad - M_\omega(10, 11, n)) q^n \pmod{11}. \end{aligned}$$

Thanks to Ramanujan's celebrated congruences (1.1), (1.2) and (1.3), we complete the proof of Corollaries 1.5, 1.6 and 1.7 by recalling (1.8).

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