An auxiliary result on the double coset $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$

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A complete set of representatives of the double coset $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$ is obtained by Radu and Sellers when N is square-free (cf. [1, Lemma 2.6]) and by Wang when $N/2$ is square-free (cf. [2, Lemma 4.3]). Here we shall give a unified proof of the following result.

Theorem A.1. *Let $N = 2^\alpha N_0$ where N_0 is an odd square-free integer. If $0 \le \alpha \le 3$, then*

$$\bigcup_{\delta|N} \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \Gamma_\infty = \Gamma. \tag{A.1}$$

Proof. Following [1], it suffices to show that

$$ch \equiv d - c/\gcd(c,N) \pmod{N/\gcd(c,N)} \tag{A.2}$$

has a solution $h \in \mathbb{Z}$ for all $c \in \mathbb{Z}$.

We first notice that (A.2) is solvable if we replace the modulus by an odd factor of $N/\gcd(c,N)$. Now let $c = 2^\beta c_0$ where c_0 is an odd integer. If $\beta \ge \alpha$, then $N/\gcd(c,N)$ has no even factors. Thanks to the Chinese remainder theorem, it suffices to show that for $0 \le \beta < \alpha$,

$$2^\beta c_0 h \equiv d - c/\gcd(c,N) \pmod{2^{\alpha-\beta}} \tag{A.3}$$

is solvable. We further observe that when $\beta = 0$, (A.3) always has a solution. Hence we assume that $0 < \beta < \alpha$.

We remark that under the assumption $0 < \beta < \alpha$, c is even and $c/\gcd(c,N)$ is odd. Furthermore, since $ad - bc = 1$, we have that d is also odd.

If $\alpha = 0$ or 1, there is no need to solve (A.3).

If $\alpha = 2$, then $\beta = 1$ is the only choice. Now (A.3) becomes

$$2c_0 h \equiv d - c/\gcd(c,N) \pmod{2},$$

which has a solution.

If $\alpha = 3$, we have $\beta = 1$ or 2. When $\beta = 1$, (A.3) becomes

$$2c_0 h \equiv d - c/\gcd(c,N) \pmod{4},$$

which is of course solvable. When $\beta = 2$, (A.3) becomes

$$4c_0 h \equiv d - c/\gcd(c,N) \pmod{2},$$

which is also solvable.

We are done! □

References

1. S. Radu and J. A. Sellers, Congruence properties modulo 5 and 7 for the pod function, *Int. J. Number Theory* **7** (2011), no. 8, 2249–2259.
2. L. Wang, Arithmetic properties of (k, ℓ)-regular bipartitions, *Bull. Aust. Math. Soc.* **95** (2017), no. 3, 353–364.

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