Multi-headed Lattice Green Function (N = 5, M = 4)
Polya Number

```
NN = 5;
MM = 4;
```

Recall some basic definitions in the paper:

$$P_{M,N}(z) := \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{1}{1 - \frac{z}{\binom{N}{M}} \sigma_M(\cos\theta_1, \ldots, \cos\theta_N)} d\theta_1 \ldots d\theta_N$$

$$R_{M,N}(z) := P_{M,N}\left(2^M \binom{N}{M} z\right) \text{ and } R_{M,N}(z) = \sum_{n \geq 0} r_{M,N}(n)\, z^n$$

Also, for M odd or $M = N$, we always have $r(2n+1) = 0$. Hence, define
$\tilde{r}_{M,N}(n) := r_{M,N}(2n)$ and $\tilde{R}_{M,N}(z) := \sum_{n \geq 0} \tilde{r}_{M,N}(n)\, z^n = \sum_{n \geq 0} r_{M,N}(2n)\, z^n$

Our goal is to find the associated Polya number of the lattice in question.

Command: UnrollRecurrence

Generate a sequence from recurrence & initial values (Koutschan’s implementation).

```
(* given a recurrence rec in f[n], compute the values {f[0],f[1],...,f[bound]}
 where inits are the initial values
 {f[0],...,f[d-1]} with d being the order of the recurrence *)
Clear[UnrollRecurrence];
UnrollRecurrence[rec1_, f_[n_], inits_, bound_] :=
  Module[{i, x, vals = inits, rec = rec1},
   If[Head[rec] =!= Equal, rec = (rec == 0)];
   rec = rec /. n → n - Max[Cases[rec, f[n + a_.] ⧴ a, Infinity]];
   Do[
    AppendTo[vals,
      Solve[rec /. n → i /. f[i] → x /. f[a_] ⧴ vals[[a + 1]], x][[1, 1, 2]]];
    , {i, Length[inits], bound}];
   Return[vals];
  ];
```

Command: SeqLimit

Compute the limit of a convergent sequence (Koutschan’s implementation).

```
(* Given the first values {f[0],...,f[m]} of a sequence f[n] and a basis of
 its asymptotic solutions, compute the limit Limit[f[n], n→Infinity]. *)
Clear[SeqLimit];
SeqLimit[data_List, asym_, n_] :=
  Module[{c, d = Length[asym], pos, ansatz, sol},
   pos = Length[data] + Range[-d, -1];
   ansatz = Array[c, d].asym;
   sol = Solve[((ansatz /. n → #) == data[[# + 1]]) & /@ pos, Array[c, d]][[1]];
   Return[N[c[d] /. sol, 200]];
  ];
```

Load RISC packages.

In[ ]:=
```
<< RISC`HolonomicFunctions`
<< RISC`Asymptotics`
<< RISC`Guess`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

--> Type ?HolonomicFunctions for help.

Asymptotics Package version 0.3
written by Manuel Kauers
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

Package GeneratingFunctions version 0.9 written by Christian Mallinger
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

Guess Package version 0.52
written by Manuel Kauers
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

In[ ]:=
```
ClearAll[Seq];
```

Load in advance the REC for $r_{4,5}(n)$ in Theorem 4.7 at the end of this file!

Translate the recurrence in term of Ore Polynomials.

In[ ]:=
```
RECinS = ToOrePolynomial[REC /. {Seq[k_] → S[α]^(k-α)}];
```

Compute the recurrence for the *partial* Green function: $\sum_{0 \le n \le n_0} r_{M,N}(n) \left(\frac{1}{2^M \binom{N}{M}}\right)^n$.

In[ ]:=
```
RECPartialGreeninS =
  DFiniteTimes[{RECinS}, Annihilator[(1/(2^MM Binomial[NN, MM]))^α, S[α]]][[1]] ** (S[α] - 1);
```

In[ ]:=
```
OrePolynomialDegree[RECPartialGreeninS, S[α]]
```

Out[ ]= 7

In[ ]:=
```
RECPartialGreen = ApplyOreOperator[RECPartialGreeninS, Seq[α]];
```

Compute the initial values of the partial Green function by the values of *r* and then generate a list.

In[∘]:=
```
RIni = {1, 0, 80, 0, 58 320, 933 120, 107 360 000, 4 032 000 000,
    305 742 850 000, 16 007 947 200 000, 1 092 754 448 110 080, 66 052 872 139 161 600};
PartialGreenIni = Table[Sum[RIni[[i]] * (1/(2^MM Binomial[NN, MM]))^(i-1), {i, 1, m}],
   {m, 0, Length@RIni}]
```

Out[∘]=
$$\left\{0,1,1,\frac{81}{80},\frac{81}{80},\frac{519129}{512000},\frac{1298187}{1280000},\frac{41558759}{40960000},\frac{20783317}{20480000},\frac{170287507149}{167772160000},\frac{170307517083}{167772160000},\frac{4258114784281293}{4194304000000000},\frac{851687461614207}{838860800000000}\right\}$$

In[∘]:=
```
Bound = 1000;

PartialGreenList = UnrollRecurrence[RECPartialGreen, Seq[α], PartialGreenIni, Bound];
```

Analyze the asymptotic behavior of the sequence of partial Green function values.

In[∘]:=
```
Asymptotics[RECPartialGreen, Seq[α]]
```

Out[∘]=
$$\left\{\frac{\left(-\frac{27}{5}\right)^{\alpha}}{\alpha^{5/2}},\frac{\left(-\frac{3}{5}\right)^{\alpha}}{\alpha^{5/2}},\frac{\left(-\frac{1}{15}\right)^{\alpha}}{\alpha^{5/2}},\frac{5^{-\alpha}}{\alpha^{9/4}},\frac{5^{-\alpha}}{\alpha^{7/4}},\frac{1}{\alpha^{3/2}},1\right\}$$

Compute the limit of partial Green function sequence and the associated Polya number.

In[∘]:=
```
lim1 = SeqLimit[PartialGreenList, Asymptotics[RECPartialGreen, Seq[α], Order → 30], α]
```

Out[∘]=
```
1.01585601716493140595083895420941527444160711020029146780423744701278215226721777263
 27943040813219853824115703511520870247046937160289062449450158487333734052073333747
 13792055872396589021016254245244
```

In[∘]:=
```
lim2 = SeqLimit[PartialGreenList, Asymptotics[RECPartialGreen, Seq[α], Order → 32], α]
```

Out[∘]=
```
1.01585601716493140595083895420941527444160711020029146780423744701278215226721777224
 06413879569487229806306569795526240438712055385554204280865374930588119333492056901 4
 84948934296220770499601164370750
```

In[∘]:=
```
lim1 - lim2
```

Out[∘]=
$$3.92152916124373262401780913371599462980833488177473485816858478355674561471858127647\,32288431215761758185214150898744 94\times 10^{-82}$$

In[∘]:=
```
1 - 1/lim2
```

Out[∘]=
```
0.01560852807584154858676912727410877062418276513066196760163352852234797898289516797
 06034133670795198963680798703151401907007725577747178531587013749873700745413863375 7
 2466649359072113579798463771783 57
```

Load the REC for $r_{4,5}(n)$ in Theorem 4.7.

In[∘]:=
```
REC =
  (2 364 822 061 925 891 270 067 722 649 600 000 + 24 311 763 241 480 737 290 507 853 496 320 000 α +
      118 884 714 388 336 585 062 289 753 767 936 000 α^2 +
      368 251 136 151 853 255 846 369 719 798 988 800 α^3 +
      811 793 640 582 985 414 140 746 797 028 474 880 α^4 +
      1 356 499 120 040 750 577 583 138 444 526 223 360 α^5 +
      1 786 835 040 377 781 128 110 811 754 937 712 640 α^6 +
      1 904 958 007 246 824 509 445 186 467 125 002 240 α^7 +
      1 674 545 402 297 600 373 785 511 713 251 000 320 α^8 +
      1 230 194 808 706 317 371 163 067 050 208 788 480 α^9 +
      762 791 807 513 049 677 466 384 009 532 538 880 α^10 +
```

$$
\begin{array}{l}
\quad 402\,079\,430\,499\,218\,110\,643\,393\,128\,200\,929\,280\,\alpha^{11} + \\
\quad 181\,085\,303\,893\,806\,582\,831\,390\,648\,576\,245\,760\,\alpha^{12} + \\
\quad 69\,909\,566\,044\,762\,687\,837\,271\,137\,604\,075\,520\,\alpha^{13} + \\
\quad 23\,174\,037\,389\,797\,607\,720\,091\,614\,796\,840\,960\,\alpha^{14} + \\
\quad 6\,597\,237\,647\,955\,223\,324\,018\,009\,760\,071\,680\,\alpha^{15} + 1\,610\,851\,715\,462\,724\,269\,782\,004\,410\,613\,760 \\
\quad\; \alpha^{16} + 336\,382\,193\,033\,012\,242\,367\,855\,858\,810\,880\,\alpha^{17} + \\
\quad 59\,795\,770\,083\,083\,316\,221\,336\,805\,703\,680\,\alpha^{18} + 8\,987\,061\,025\,545\,721\,077\,834\,511\,810\,560\,\alpha^{19} + \\
\quad 1\,131\,237\,375\,988\,193\,565\,613\,353\,861\,120\,\alpha^{20} + 117\,704\,523\,870\,056\,936\,584\,154\,972\,160\,\alpha^{21} + \\
\quad 9\,941\,030\,662\,497\,120\,749\,554\,237\,440\,\alpha^{22} + 664\,040\,244\,922\,741\,425\,721\,835\,520\,\alpha^{23} + \\
\quad 33\,746\,986\,442\,943\,554\,031\,452\,160\,\alpha^{24} + 1\,225\,566\,587\,608\,656\,091\,545\,600\,\alpha^{25} + \\
\quad 28\,320\,365\,528\,012\,449\,382\,400\,\alpha^{26} + 312\,808\,771\,118\,086\,225\,920\,\alpha^{27}\big)\ \mathrm{Seq}[\alpha] + \\
\big(880\,540\,948\,213\,763\,261\,498\,004\,602\,880\,000 + 8\,086\,612\,414\,279\,581\,582\,690\,097\,299\,456\,000\,\alpha + \\
\quad 35\,535\,843\,625\,080\,580\,938\,628\,852\,403\,404\,800\,\alpha^{2} + \\
\quad 99\,482\,199\,073\,846\,865\,130\,149\,987\,053\,731\,840\,\alpha^{3} + \\
\quad 199\,278\,215\,238\,194\,877\,084\,174\,219\,759\,058\,944\,\alpha^{4} + \\
\quad 304\,147\,288\,569\,704\,121\,767\,283\,668\,058\,636\,288\,\alpha^{5} + \\
\quad 367\,726\,422\,460\,034\,552\,713\,877\,456\,306\,307\,072\,\alpha^{6} + \\
\quad 361\,508\,986\,147\,801\,089\,153\,130\,211\,095\,805\,952\,\alpha^{7} + \\
\quad 294\,331\,319\,744\,750\,632\,422\,172\,167\,712\,997\,376\,\alpha^{8} + \\
\quad 201\,108\,607\,972\,501\,732\,293\,906\,606\,562\,934\,784\,\alpha^{9} + \\
\quad 116\,437\,788\,942\,848\,727\,536\,075\,769\,222\,856\,704\,\alpha^{10} + \\
\quad 57\,524\,299\,296\,878\,619\,402\,424\,939\,339\,382\,784\,\alpha^{11} + \\
\quad 24\,367\,165\,878\,769\,872\,656\,509\,536\,747\,061\,248\,\alpha^{12} + \\
\quad 8\,877\,402\,295\,660\,764\,714\,512\,245\,808\,234\,496\,\alpha^{13} + \\
\quad 2\,785\,748\,984\,068\,408\,698\,625\,918\,477\,467\,648\,\alpha^{14} + \\
\quad 752\,972\,653\,647\,501\,430\,958\,086\,738\,673\,664\,\alpha^{15} + 175\,049\,743\,314\,674\,169\,771\,167\,299\,534\,848 \\
\quad\; \alpha^{16} + 34\,895\,534\,864\,837\,208\,484\,258\,292\,957\,184\,\alpha^{17} + \\
\quad 5\,936\,277\,532\,573\,962\,980\,718\,997\,929\,984\,\alpha^{18} + 855\,818\,515\,821\,739\,179\,539\,429\,326\,848\,\alpha^{19} + \\
\quad 103\,560\,073\,600\,267\,246\,364\,541\,321\,216\,\alpha^{20} + 10\,380\,185\,487\,431\,012\,018\,005\,475\,328\,\alpha^{21} + \\
\quad 846\,180\,664\,706\,397\,472\,693\,420\,032\,\alpha^{22} + 54\,656\,640\,176\,185\,180\,963\,209\,216\,\alpha^{23} + \\
\quad 2\,690\,612\,916\,385\,314\,156\,576\,768\,\alpha^{24} + 94\,804\,345\,329\,795\,433\,758\,720\,\alpha^{25} + \\
\quad 2\,128\,785\,749\,082\,227\,343\,360\,\alpha^{26} + 22\,881\,382\,331\,785\,936\,896\,\alpha^{27}\big)\ \mathrm{Seq}[1+\alpha] + \\
\big(-664\,078\,540\,666\,702\,251\,488\,371\,015\,680\,000 - 5\,805\,956\,958\,011\,506\,960\,041\,778\,348\,032\,000\,\alpha - \\
\quad 24\,298\,272\,789\,380\,152\,495\,188\,221\,126\,246\,400\,\alpha^{2} - \\
\quad 64\,810\,405\,629\,301\,547\,428\,216\,819\,254\,558\,720\,\alpha^{3} - \\
\quad 123\,755\,374\,367\,469\,269\,296\,809\,845\,353\,611\,264\,\alpha^{4} - \\
\quad 180\,149\,375\,502\,996\,189\,202\,275\,648\,542\,982\,144\,\alpha^{5} - \\
\quad 207\,865\,771\,244\,125\,682\,287\,781\,841\,861\,722\,112\,\alpha^{6} - \\
\quad 195\,153\,222\,041\,523\,657\,876\,484\,723\,267\,989\,504\,\alpha^{7} - \\
\quad 151\,846\,270\,858\,495\,120\,363\,896\,477\,860\,167\,680\,\alpha^{8} - \\
\quad 99\,230\,231\,828\,276\,421\,932\,960\,434\,682\,314\,752\,\alpha^{9} - \\
\quad 54\,993\,115\,047\,787\,497\,911\,079\,580\,675\,899\,392\,\alpha^{10} - \\
\quad 26\,028\,017\,908\,489\,825\,928\,212\,462\,245\,453\,824\,\alpha^{11} - \\
\quad 10\,572\,113\,416\,646\,586\,933\,511\,582\,698\,766\,336\,\alpha^{12} - \\
\quad 3\,696\,722\,231\,163\,815\,760\,173\,082\,026\,344\,448\,\alpha^{13} - 1\,114\,468\,173\,061\,041\,282\,670\,805\,399\,093\,248 \\
\quad\; \alpha^{14} - 289\,688\,969\,845\,746\,113\,335\,461\,572\,931\,584\,\alpha^{15} - \\
\quad 64\,831\,091\,647\,102\,802\,811\,533\,842\,055\,168\,\alpha^{16} - 12\,454\,053\,009\,083\,005\,771\,527\,566\,163\,968\,\alpha^{17} - \\
\quad 2\,043\,760\,292\,966\,696\,499\,523\,264\,184\,320\,\alpha^{18} - 284\,532\,912\,366\,921\,324\,027\,166\,588\,928\,\alpha^{19} - \\
\quad 33\,284\,416\,956\,384\,385\,896\,458\,223\,616\,\alpha^{20} - 3\,228\,606\,478\,351\,534\,833\,828\,626\,432\,\alpha^{21} - \\
\quad 254\,974\,947\,491\,313\,890\,128\,560\,128\,\alpha^{22} - 15\,972\,126\,457\,377\,261\,067\,698\,176\,\alpha^{23} - \\
\quad 763\,333\,007\,662\,980\,725\,211\,136\,\alpha^{24} - 26\,138\,887\,552\,462\,651\,129\,856\,\alpha^{25} - \\
\quad 570\,997\,443\,951\,748\,710\,400\,\alpha^{26} - 5\,976\,795\,675\,008\,958\,464\,\alpha^{27}\big)\ \mathrm{Seq}[2+\alpha] + \\
\big(36\,337\,840\,931\,616\,555\,318\,702\,833\,664\,000 + 310\,343\,693\,247\,202\,072\,877\,171\,431\,833\,600\,\alpha + \\
\quad 1\,268\,062\,726\,217\,635\,641\,408\,454\,051\,430\,400\,\alpha^{2} + 3\,300\,521\,955\,790\,071\,740\,463\,976\,232\,263\,680 \\
\quad\; \alpha^{3} + 6\,146\,984\,578\,367\,464\,065\,862\,054\,879\,242\,240\,\alpha^{4} +
\end{array}
$$

8 723 512 529 514 925 026 222 139 080 468 480 α^5 + 9 808 817 646 565 897 068 529 809 213 239 808
α^6 + 8 970 447 157 798 999 809 214 350 039 412 224 α^7 +
6 796 618 106 855 403 262 931 535 421 469 184 α^8 + 4 323 600 610 674 086 572 350 145 316 732 416
α^9 + 2 331 860 127 398 843 166 087 931 718 971 904 α^{10} +
1 073 804 990 271 736 796 663 841 511 156 224 α^{11} + 424 279 297 446 148 516 898 147 199 947 264
α^{12} + 144 293 344 557 135 741 340 883 292 465 664 α^{13} +
42 304 696 119 152 808 149 756 544 291 840 α^{14} + 10 693 366 157 119 575 923 154 101 714 944 α^{15} +
2 327 102 570 668 214 059 453 238 664 192 α^{16} + 434 708 874 971 795 823 099 840 116 736 α^{17} +
69 373 988 097 051 870 247 906 934 784 α^{18} + 9 393 304 762 567 159 143 035 764 736 α^{19} +
1 068 815 757 774 279 757 481 902 080 α^{20} + 100 861 570 825 855 881 262 923 776 α^{21} +
7 750 770 733 439 394 600 976 384 α^{22} + 472 551 963 878 997 639 561 216 α^{23} +
21 986 541 883 647 884 001 280 α^{24} + 733 188 729 988 561 502 208 α^{25} +
15 602 375 112 618 147 840 α^{26} + 159 149 910 074 064 896 α^{27}) Seq[3 + α] +
(1 737 772 868 400 007 324 872 130 560 000 + 14 528 609 204 414 291 845 066 255 564 800 α +
58 083 087 258 852 534 411 685 975 019 520 α^2 + 147 846 850 915 658 722 383 612 355 430 400 α^3 +
269 164 023 324 400 460 962 054 275 740 928 α^4 + 373 240 816 513 597 979 905 593 440 661 888 α^5 +
409 908 879 949 766 514 326 399 060 864 064 α^6 + 366 016 393 873 249 701 940 597 734 061 344 α^7 +
270 676 671 846 416 971 917 873 052 917 920 α^8 + 168 013 318 310 785 666 403 759 927 887 584 α^9 +
88 393 926 598 940 439 065 183 725 045 600 α^{10} + 39 697 363 634 496 672 642 069 844 386 912 α^{11} +
15 293 672 611 896 263 618 803 193 519 136 α^{12} + 5 070 491 874 452 377 148 797 920 831 072 α^{13} +
1 449 002 022 519 967 409 403 051 116 512 α^{14} + 356 957 682 436 813 381 749 659 746 304 α^{15} +
75 700 244 148 872 939 301 421 992 640 α^{16} + 13 779 371 789 456 905 170 877 563 840 α^{17} +
2 142 685 081 818 193 152 012 367 872 α^{18} + 282 685 926 147 777 894 282 083 328 α^{19} +
31 341 335 886 140 485 043 322 880 α^{20} + 2 881 942 426 887 984 021 438 464 α^{21} +
215 812 414 752 103 173 455 872 α^{22} + 12 823 036 513 484 289 343 488 α^{23} +
581 508 878 853 457 575 936 α^{24} + 18 903 053 117 719 314 432 α^{25} +
392 186 219 850 629 120 α^{26} + 3 900 964 176 134 144 α^{27}) Seq[4 + α] +
(– 36 446 102 109 669 030 849 285 120 000 – 301 794 930 778 773 719 063 321 856 000 α –
1 194 401 836 156 084 887 609 064 224 000 α^2 – 3 008 156 975 709 477 795 289 491 275 520 α^3 –
5 415 770 546 395 539 670 222 530 489 360 α^4 – 7 422 453 554 874 065 600 190 474 289 032 α^5 –
8 052 206 383 842 449 223 124 682 104 644 α^6 – 7 098 162 826 794 167 361 280 152 144 294 α^7 –
5 179 144 111 408 801 590 076 035 892 950 α^8 – 3 169 950 795 733 038 711 522 140 215 280 α^9 –
1 643 499 248 947 095 475 104 215 404 004 α^{10} – 726 910 788 718 026 537 302 273 862 144 α^{11} –
275 635 972 025 251 416 199 969 761 656 α^{12} – 89 889 728 147 001 421 773 544 625 132 α^{13} –
25 251 994 806 501 150 584 061 125 784 α^{14} – 6 111 409 098 652 595 993 659 452 026 α^{15} –
1 272 483 225 563 071 816 917 699 490 α^{16} – 227 273 250 419 552 627 170 585 084 α^{17} –
34 655 941 701 831 856 557 922 624 α^{18} – 4 480 880 404 407 427 210 024 320 α^{19} –
486 585 842 769 876 461 484 032 α^{20} – 43 798 304 089 562 788 663 296 α^{21} –
3 208 710 131 027 557 023 744 α^{22} – 186 416 522 833 559 945 216 α^{23} –
8 261 380 192 874 790 912 α^{24} – 262 301 388 296 421 376 α^{25} –
5 312 632 953 241 600 α^{26} – 51 561 082 388 480 α^{27}) Seq[5 + α] +
(– 154 404 486 709 237 819 219 968 000 – 1 265 327 918 255 018 927 110 348 800 α –
4 953 641 658 930 095 511 385 751 040 α^2 – 12 335 446 851 783 544 166 937 390 720 α^3 –
21 947 702 123 383 074 616 990 244 544 α^4 – 29 712 684 443 300 038 100 072 561 760 α^5 –
31 824 626 177 807 101 870 129 360 368 α^6 – 27 684 339 638 906 598 652 692 786 888 α^7 –
19 923 668 408 873 674 929 361 243 572 α^8 – 12 021 754 897 932 453 908 473 126 194 α^9 –
6 141 402 912 303 808 338 721 284 327 α^{10} – 2 675 090 519 652 464 763 702 625 995 α^{11} –
998 451 712 547 824 111 144 656 513 α^{12} – 320 337 381 856 256 276 567 115 789 α^{13} –
88 485 146 094 830 787 771 471 525 α^{14} – 21 045 641 782 461 353 200 898 049 α^{15} –
4 304 140 182 149 530 399 276 227 α^{16} – 754 678 659 252 915 954 749 073 α^{17} –
112 910 766 050 133 819 763 020 α^{18} – 14 316 213 223 182 938 203 068 α^{19} –
1 523 679 350 645 560 062 336 α^{20} – 134 345 128 624 663 841 280 α^{21} –
9 635 762 018 738 626 560 α^{22} – 547 760 583 383 666 688 α^{23} – 23 739 371 943 886 848 α^{24} –
736 693 272 182 784 α^{25} – 14 575 541 944 320 α^{26} – 138 110 042 112 α^{27}) Seq[6 + α];