Multi-headed Lattice Green Function (N = 4, M = 3)

Polya Number

In[∘]:=
```
NN = 4;
MM = 3;
```

Recall some basic definitions in the paper:

$$P_{M,N}(z) := \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{1}{1 - \frac{z}{\binom{N}{M}} \sigma_M(\cos\theta_1, \ldots, \cos\theta_N)} d\theta_1 \ldots d\theta_N$$

$$R_{M,N}(z) := P_{M,N}\left(2^M \binom{N}{M} z\right) \text{ and } R_{M,N}(z) = \sum_{n \geq 0} r_{M,N}(n)\, z^n$$

Also, for M odd or $M = N$, we always have $r(2n+1) = 0$. Hence, define

$\tilde{r}_{M,N}(n) := r_{M,N}(2n)$ and $\tilde{R}_{M,N}(z) := \sum_{n \geq 0} \tilde{r}_{M,N}(n)\, z^n = \sum_{n \geq 0} r_{M,N}(2n)\, z^n$

Our goal is to find the associated Polya number of the lattice in question.

Command: UnrollRecurrence

Generate a sequence from recurrence & initial values (Koutschan's implementation).

In[∘]:=
```
(* given a recurrence rec in f[n], compute the values {f[0],f[1],...,f[bound]}
 where inits are the initial values
 {f[0],...,f[d-1]} with d being the order of the recurrence *)
Clear[UnrollRecurrence];
UnrollRecurrence[rec1_, f_[n_], inits_, bound_] :=
  Module[{i, x, vals = inits, rec = rec1},
   If[Head[rec] =!= Equal, rec = (rec == 0)];
   rec = rec /. n → n - Max[Cases[rec, f[n + a_.] :→ a, Infinity]];
   Do[
    AppendTo[vals,
      Solve[rec /. n → i /. f[i] → x /. f[a_] :→ vals[[a + 1]], x][[1, 1, 2]]];
    , {i, Length[inits], bound}];
   Return[vals];
  ];
```

Command: SeqLimit

Compute the limit of a convergent sequence (Koutschan's implementation).

In[∘]:=
```
(* Given the first values {f[0],...,f[m]} of a sequence f[n] and a basis of
 its asymptotic solutions, compute the limit Limit[f[n], n→Infinity]. *)
Clear[SeqLimit];
SeqLimit[data_List, asym_, n_] :=
  Module[{c, d = Length[asym], pos, ansatz, sol},
   pos = Length[data] + Range[-d, -1];
   ansatz = Array[c, d].asym;
   sol = Solve[((ansatz /. n → #) == data[[# + 1]]) & /@ pos, Array[c, d]][[1]];
   Return[N[c[d] /. sol, 200]];
  ];
```

Load RISC packages.

In[∘]:=
```
<< RISC`HolonomicFunctions`
<< RISC`Asymptotics`
<< RISC`Guess`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan


```
--> Type  ?HolonomicFunctions  for help.
```

Asymptotics Package version 0.3
written by Manuel Kauers

Package GeneratingFunctions version 0.9 written by Christian Mallinger

Guess Package version 0.52
written by Manuel Kauers

In[∘]:=
```
ClearAll[Seq];
```

Load in advance the REC for $\tilde{r}_{3,4}(n)$ in Theorem 4.3 at the end of this file!

Translate the recurrence in term of Ore Polynomials.

In[∘]:=
```
RECinS = ToOrePolynomial[REC /. {Seq[k_] → S[α]^(k-α)}];
```

Compute the recurrence for the *partial* Green function: $\sum_{0\leq n\leq n_0} \tilde{r}_{M,N}(n)\left(\frac{1}{2^M\binom{N}{M}}\right)^{2n}$.

In[∘]:=
```
RECPartialGreeninS =
  DFiniteTimes[{RECinS}, Annihilator[(1/(2^MM Binomial[NN, MM]))^(2 α), S[α]]][[1]] **
   (S[α] - 1);
```

In[∘]:=
```
OrePolynomialDegree[RECPartialGreeninS, S[α]]
```

Out[∘]= 5

In[∘]:=
```
RECPartialGreen = ApplyOreOperator[RECPartialGreeninS, Seq[α]];
```

Compute the initial values of the partial Green function by the values of $\tilde{r}$ and then generate a list.

In[ ]:=
```
RIni = {1, 32, 6048, 2451200, 1391236000, 921422380032, 663895856219904};
PartialGreenIni =
  Table[Sum[RIni[[i]] * (1/(2^MM Binomial[NN, MM]))^(2 (i-1)), {i, 1, m}], {m, 0, Length@RIni}]
```

Out[ ]= $\left\{0, 1, \frac{33}{32}, \frac{33981}{32768}, \frac{4359143}{4194304}, \frac{35753575581}{34359738368}, \frac{1145014245135}{1099511627776}, \frac{4692571691261319}{4503599627370496}\right\}$

In[ ]:=
```
Bound = 1000;

PartialGreenList = UnrollRecurrence[RECPartialGreen, Seq[α], PartialGreenIni, Bound];
```

Analyze the asymptotic behavior of the sequence of partial Green function values.

In[ ]:=
```
Asymptotics[RECPartialGreen, Seq[α]]
```

Out[ ]= $\left\{\frac{64^{-\alpha}}{\alpha^2}, \frac{4^{-\alpha}}{\alpha^2}, \frac{1}{\alpha^2}, \frac{1}{\alpha}, 1\right\}$

Compute the limit of partial Green function sequence and the associated Polya number.

In[ ]:=
```
lim1 = SeqLimit[PartialGreenList, Asymptotics[RECPartialGreen, Seq[α], Order → 30], α]
```

Out[ ]=
```
1.0452879180865911417843270133824978673752777397256790765800746729908972504362795260587280053325719131916418189882225607533866047201082307920379467818546491857995110796787292822423937716338115597824133
```

In[ ]:=
```
lim2 = SeqLimit[PartialGreenList, Asymptotics[RECPartialGreen, Seq[α], Order → 32], α]
```

Out[ ]=
```
1.0452879180865911417843270133824978673752777397256790765800746729908978874655508327043747385428792555875743875662020296967016288176778352654553918272341499670218733469078853054713472431146366985461419
```

In[ ]:=
```
lim1 - lim2
```

Out[ ]=
```
-6.370292713066456467332103073423959325685779794689433150240975696044734174450453795007812223622672291560232289534714808251387637287 × 10^-70
```

In[ ]:=
```
1 - 1/lim2
```

Out[ ]=
```
0.04332578355013523911985002959583436982621694420764581838291342263347221652221583425454986885705193317718084864475491816063041260745264175452555003996602296568042967176528834796305034315419700621172431
```

Load the REC for $\tilde{r}_{3,4}(n)$ in Theorem 4.3.

```
In[∘]:= REC = (221 086 792 032 258 663 383 040 +
    3 002 581 182 281 579 476 549 632 α + 18 896 284 453 973 181 469 818 880 α^2 +
    73 337 056 136 834 742 984 114 176 α^3 + 197 017 275 538 043 925 583 364 096 α^4 +
    389 745 626 428 476 129 286 291 456 α^5 + 589 529 476 016 351 811 509 157 888 α^6 +
    698 690 177 713 813 455 561 031 680 α^7 + 659 396 154 092 196 671 988 432 896 α^8 +
    500 766 687 956 261 350 615 810 048 α^9 + 307 887 490 552 535 839 569 608 704 α^10 +
    153 616 793 330 862 792 246 296 576 α^11 + 62 125 104 506 185 984 379 977 728 α^12 +
    20 265 270 278 609 884 774 662 144 α^13 + 5 282 843 409 745 454 510 899 200 α^14 +
    1 084 193 901 809 507 676 192 768 α^15 + 171 154 981 038 855 165 050 880 α^16 +
    20 040 031 539 432 857 272 320 α^17 + 1 638 003 152 561 664 688 128 α^18 +
    83 373 097 696 100 352 000 α^19 + 1 988 330 027 074 191 360 α^20) Seq[α] +
  (-123 596 648 884 357 621 088 256 - 1 387 410 081 329 207 115 251 712 α -
    7 308 010 505 383 031 273 947 136 α^2 - 24 020 604 752 075 269 740 691 456 α^3 -
    55 262 591 055 735 725 773 815 808 α^4 - 94 607 549 345 038 165 436 006 400 α^5 -
    125 070 786 847 359 746 869 821 440 α^6 - 130 760 992 638 503 780 446 109 696 α^7 -
    109 819 712 522 499 293 630 693 376 α^8 - 74 830 049 897 678 615 099 736 064 α^9 -
    41 599 115 200 046 517 939 601 408 α^10 - 18 902 277 196 351 684 209 803 264 α^11 -
    7 008 965 526 989 775 347 122 176 α^12 - 2 109 519 207 312 665 281 560 576 α^13 -
    510 375 764 108 304 797 663 232 α^14 - 97 744 104 267 386 959 429 632 α^15 -
    14 472 279 363 085 494 386 688 α^16 - 1 596 811 738 769 963 089 920 α^17 -
    123 530 156 260 699 668 480 α^18 - 5 975 058 303 292 538 880 α^19 - 135 920 997 944 524 800 α^20)
   Seq[1 + α] + (2 413 729 498 666 800 513 024 + 25 435 086 835 865 925 058 560 α +
    125 542 481 225 411 227 975 680 α^2 + 386 097 946 352 750 392 590 336 α^3 +
    830 183 396 028 360 968 208 384 α^4 + 1 327 255 653 860 270 011 465 728 α^5 +
    1 637 850 112 836 596 110 688 256 α^6 + 1 598 197 760 043 557 807 628 288 α^7 +
    1 252 980 911 862 994 173 739 008 α^8 + 797 358 770 338 813 407 952 896 α^9 +
    414 276 959 391 975 941 603 328 α^10 + 176 103 421 096 866 815 410 176 α^11 +
    61 159 515 859 482 838 548 480 α^12 + 17 263 930 413 062 410 149 888 α^13 +
    3 923 295 133 237 310 914 560 α^14 + 706 924 713 366 338 125 824 α^15 +
    98 652 029 401 005 981 696 α^16 + 10 278 087 291 823 325 184 α^17 + 752 234 327 699 226 624 α^18 +
    34 490 272 274 841 600 α^19 + 745 214 176 788 480 α^20) Seq[2 + α] +
  (-9 569 617 440 812 835 840 - 97 443 791 378 162 009 856 α - 463 583 339 186 644 316 800 α^2 -
    1 370 837 922 368 778 354 176 α^3 - 2 827 452 328 200 593 850 560 α^4 -
    4 326 575 055 112 730 856 640 α^5 - 5 099 519 612 920 329 528 000 α^6 -
    4 743 666 552 937 883 189 952 α^7 - 3 539 068 890 050 114 722 112 α^8 -
    2 139 750 587 880 300 657 856 α^9 - 1 054 730 779 373 468 537 920 α^10 -
    424 824 967 934 147 228 480 α^11 - 139 643 546 214 642 867 648 α^12 -
    37 274 084 807 088 072 384 α^13 - 8 003 802 897 605 020 608 α^14 -
    1 361 866 764 260 304 576 α^15 - 179 386 646 751 384 192 α^16 - 17 635 678 788 631 680 α^17 -
    1 217 772 669 657 600 α^18 - 52 679 537 809 920 α^19 - 1 074 030 451 200 α^20) Seq[3 + α] +
  (9 051 531 325 562 880 + 90 332 029 095 081 984 α + 420 333 410 362 428 416 α^2 +
    1 213 206 945 955 473 664 α^3 + 2 437 377 188 874 087 136 α^4 + 3 625 291 113 645 770 712 α^5 +
    4 144 688 219 837 114 384 α^6 + 3 731 957 019 300 871 994 α^7 + 2 689 507 840 271 682 912 α^8 +
    1 567 534 832 320 365 967 α^9 + 743 334 125 295 350 476 α^10 + 287 455 002 784 035 524 α^11 +
    90 539 774 552 500 272 α^12 + 23 112 095 925 472 389 α^13 + 4 737 102 973 509 780 α^14 +
    767 930 664 461 310 α^15 + 96 195 146 877 576 α^16 + 8 977 485 504 456 α^17 +
    587 451 930 408 α^18 + 24 041 253 600 α^19 + 462 944 160 α^20) Seq[4 + α];
```