Multi-headed Lattice Green Function (N = 5, M = 4)

In[∘]:=
```
NN = 5;
MM = 4;
```

Recall some basic definitions in the paper:

$$P_{M,N}(z) := \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{1}{1 - \frac{z}{\binom{N}{M}} \sigma_M(\cos\theta_1, \ldots, \cos\theta_N)} d\theta_1 \ldots d\theta_N$$

$R_{M,N}(z) := P_{M,N}\left(2^M \binom{N}{M} z\right)$ and $R_{M,N}(z) = \sum_{n \geq 0} r_{M,N}(n) z^n$

Also, for M odd or $M = N$, we always have $r(2n+1) = 0$. Hence, define
$\tilde{r}_{M,N}(n) := r_{M,N}(2n)$ and $\tilde{R}_{M,N}(z) := \sum_{n \geq 0} \tilde{r}_{M,N}(n) z^n = \sum_{n \geq 0} r_{M,N}(2n) z^n$

Our goal is to find:

Case 1. *M* even and *M* ≠ *N*:
- recurrences (REC) for $r(n)$ or differential equations (ODE) for $R(z)$.

Case 2. *M* odd or *M* = *N*:
- recurrences (REC) for $\tilde{r}(n)$ or differential equations (ODE) for $\tilde{R}(z)$.

Command: UnrollRecurrence

Generate a sequence from recurrence & initial values (Koutschan's implementation).

In[∘]:=
```
(* given a recurrence rec in f[n], compute the values {f[0],f[1],...,f[bound]}
 where inits are the initial values
 {f[0],...,f[d-1]} with d being the order of the recurrence *)
Clear[UnrollRecurrence];
UnrollRecurrence[rec1_, f_[n_], inits_, bound_] :=
  Module[{i, x, vals = inits, rec = rec1},
   If[Head[rec] =!= Equal, rec = (rec == 0)];
   rec = rec /. n → n - Max[Cases[rec, f[n + a_.] :→ a, Infinity]];
   Do[
    AppendTo[vals,
      Solve[rec /. n → i /. f[i] → x /. f[a_] :→ vals[[a + 1]], x][[1, 1, 2]]];
    , {i, Length[inits], bound}];
   Return[vals];
  ];
```

Load RISC packages.

In[∘]:=
```
<< RISC`HolonomicFunctions`
<< RISC`Asymptotics`
<< RISC`Guess`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan

```
--> Type  ?HolonomicFunctions  for help.
```

Asymptotics Package version 0.3
written by Manuel Kauers

Package GeneratingFunctions version 0.9 written by Christian Mallinger

Guess Package version 0.52
written by Manuel Kauers

Apply creative telescoping to the even-indexed subsequence $\tilde{r}_e(n) := r(2\,n)$.

```
In[•]:= ClearAll[k1, k2, k3, k4, k5, z, w, α, β];

In[•]:= k5 = α - k1 - k2 - k3 - k4;
summandEVEN = Binomial[2 α, 2 k1] Binomial[2 α - 2 k1, 2 k2]
   Binomial[2 α - 2 k1 - 2 k2, 2 k3] Binomial[2 α - 2 k1 - 2 k2 - 2 k3, 2 k4]
   Binomial[2 (α - k1), α - k1] Binomial[2 (α - k2), α - k2] Binomial[2 (α - k3), α - k3]
   Binomial[2 (α - k4), α - k4] Binomial[2 (α - k5), α - k5];

In[•]:= Timing[ann0EVEN = Annihilator[summandEVEN, {S[k1], S[k2], S[k3], S[k4], S[α]}];]
Out[•]= {0.078125, Null}

In[•]:= Timing[ann1EVEN = FindCreativeTelescoping[ann0EVEN, S[k1] - 1][[1]];]
Out[•]= {433.984, Null}

In[•]:= Timing[ann2EVEN = FindCreativeTelescoping[ann1EVEN, S[k2] - 1][[1]];]
Out[•]= {12354.5, Null}

In[•]:= Timing[ann3EVEN = FindCreativeTelescoping[ann2EVEN, S[k3] - 1][[1]];]
Out[•]= {39765., Null}

In[•]:= Timing[ann4EVEN = FindCreativeTelescoping[ann3EVEN, S[k4] - 1][[1]];]
Out[•]= {44146.1, Null}
```

Alternatively, you may import the value of {ann1EVEN, ..., ann4EVEN} from an external file.

```
In[•]:= {ann1EVEN, ann2EVEN, ann3EVEN, ann4EVEN} =
  ToExpression[Import[NotebookDirectory[] <> "Data-N5M4-Sum-EVEN.txt"]];
```

ann4EVEN gives a REC for $\tilde{r}_e(n)$.

Apply creative telescoping to the odd-indexed subsequence $\tilde{r}_o(n) := r(2\,n+1)$.

```
In[•]:= ClearAll[k1, k2, k3, k4, k5, z, w, α, β];

In[•]:= k5 = α + (1 - NN)/2 - k1 - k2 - k3 - k4;
        summandODD = Binomial[2 α + 1, 2 k1 + 1] Binomial[(2 α + 1) - (2 k1 + 1), 2 k2 + 1]
            Binomial[(2 α + 1) - (2 k1 + 1) - (2 k2 + 1), 2 k3 + 1]
            Binomial[(2 α + 1) - (2 k1 + 1) - (2 k2 + 1) - (2 k3 + 1), 2 k4 + 1]
            Binomial[2 (α - k1), α - k1] Binomial[2 (α - k2), α - k2] Binomial[2 (α - k3), α - k3]
            Binomial[2 (α - k4), α - k4] Binomial[2 (α - k5), α - k5];

In[•]:= Timing[ann0ODD = Annihilator[summandODD, {S[k1], S[k2], S[k3], S[k4], S[α]}];]
Out[•]= {0.09375, Null}

In[•]:= Timing[ann1ODD = FindCreativeTelescoping[ann0ODD, S[k1] - 1][[1]];]
Out[•]= {419.172, Null}

In[•]:= Timing[ann2ODD = FindCreativeTelescoping[ann1ODD, S[k2] - 1][[1]];]
Out[•]= {15 208.2, Null}

In[•]:= Timing[ann3ODD = FindCreativeTelescoping[ann2ODD, S[k3] - 1][[1]];]
Out[•]= {35 861.1, Null}

In[•]:= Timing[ann4ODD = FindCreativeTelescoping[ann3ODD, S[k4] - 1][[1]];]
Out[•]= {42 672., Null}
```

Alternatively, you may import the value of {ann1ODD, ..., ann4ODD} from an external file.

```
In[•]:= {ann1ODD, ann2ODD, ann3ODD, ann4ODD} =
          ToExpression[Import[NotebookDirectory[] <> "Data-N5M4-Sum-ODD.txt"]];
```

ann4ODD gives a REC for $\tilde{r}_o(n)$.

Compute the REC for *r*(*n*).

REC: Order 12

ODE: Order 83, Degree 12

We first store the RECs for $\tilde{r}_e(n)$ and $\tilde{r}_o(n)$.

```
In[•]:= RECNormalizedinSEVEN = ann4EVEN[[1]];
        RECNormalizedinSOrderEVEN = OrePolynomialDegree[RECNormalizedinSEVEN]
Out[•]= 6

In[•]:= RECNormalizedinSODD = ann4ODD[[1]];
        RECNormalizedinSOrderODD = OrePolynomialDegree[RECNormalizedinSODD]
Out[•]= 6
```

Then we derive the RECs for sequences
$\{r(0),\ 0,\ r(2),\ 0,\ \ldots\}$ and
$\{0,\ r(1),\ 0,\ r(3),\ \ldots\}$,
and compute the REC for their linear combination, including
$\{r(0),\ 0,\ r(2),\ 0,\ \ldots\} + \{0,\ r(1),\ 0,\ r(3),\ \ldots\} = \{r(0),\ r(1),\ r(2),\ r(3),\ \ldots\}$.

```
In[•]:= RECNormalizedEVENnew =
          OrePolynomialSubstitute[{RECNormalizedinSEVEN}, {α → (α - 0) / 2, S[α] → S[α]^2}];
```

In[ ]:=
```
RECNormalizedODDnew =
  OrePolynomialSubstitute[{RECNormalizedinSODD}, {α → (α - 1) / 2, S[α] → S[α]^2}];
```

In[ ]:=
```
RECNormalizedinS = DFinitePlus[RECNormalizedEVENnew, RECNormalizedODDnew][[1]];
```

In[ ]:=
```
RECNormalizedinSOrder = OrePolynomialDegree[RECNormalizedinS]
```

Out[ ]= 12

In[ ]:=
```
ODENormalizedinD =
  NormalizeCoefficients[DFiniteRE2DE[{RECNormalizedinS}, {α}, {w}][[1]]];
```

In[ ]:=
```
ODENormalizedinTheta =
  NormalizeCoefficients[ChangeOreAlgebra[w ** ODENormalizedinD, OreAlgebra[Euler[w]]]];
```

In[ ]:=
```
ODENormalizedinThetaOrder = OrePolynomialDegree[ODENormalizedinTheta, Euler[w]]
```

Out[ ]= 83

In[ ]:=
```
ODENormalizedinThetaDegree =
 Max[Exponent[OrePolynomialListCoefficients[ODENormalizedinTheta], w]]
```

Out[ ]= 12

We also write this REC explicitly.

In[ ]:=
```
ClearAll[Seq];
SeqNormalized = ApplyOreOperator[RECNormalizedinS, Seq[α]];
```

The initial values of $r(n)$ are as follows.

In[ ]:=
```
SeqListIni = {};

MAX = 20;

For[n = 0, n ≤ MAX, n++,
  coord = Select[Tuples[Table[i, {i, 0, n}], NN], Total[#] == n &];
  size = Length@coord;
  p = Sum[Multinomial[Sequence @@ (2 coord[[i]])] * Product[
      Binomial[2 n - 2 coord[[i, j]], n - coord[[i, j]]], {j, 1, NN}], {i, 1, size}];
  SeqListIni = Append[SeqListIni, p];

  coord = Select[Tuples[Table[i, {i, 0, n}], NN], Total[#] == n + (1 - NN) / 2 &];
  size = Length@coord;
  p = Sum[Multinomial[Sequence @@ (2 coord[[i]] + 1)] * Product[
      Binomial[2 n - 2 coord[[i, j]], n - coord[[i, j]]], {j, 1, NN}], {i, 1, size}];
  SeqListIni = Append[SeqListIni, p];
 ];

SeqListIni

seq[n_] := SeqListIni[[n + 1]];
```

```
Out[•]= {1, 0, 80, 0, 58 320, 933 120, 107 360 000, 4 032 000 000, 305 742 850 000, 16 007 947 200 000,
 1 092 754 448 110 080, 66 052 872 139 161 600, 4 433 464 272 394 080 000, 287 105 556 124 600 012 800,
 19 441 756 158 387 587 481 600, 1 307 659 624 636 945 150 771 200, 89 869 341 860 254 106 893 314 000,
 6 191 536 013 119 541 254 794 624 000, 431 788 153 780 445 031 117 712 736 000,
 30 259 578 124 053 738 011 950 295 040 000, 2 137 643 722 042 861 014 846 923 875 678 720,
 151 778 757 062 056 398 402 787 590 848 716 800, 10 840 750 037 089 338 687 405 094 405 540 454 400,
 777 883 218 982 271 229 558 388 389 382 825 574 400,
 56 080 935 388 938 320 492 345 601 400 578 969 030 400,
 4 059 518 371 465 289 501 011 809 299 957 269 579 653 120,
 295 006 495 123 163 326 450 011 592 999 699 774 386 176 000,
 21 513 746 057 744 924 699 009 848 676 027 694 742 870 425 600,
 1 574 148 924 348 897 968 127 657 314 112 417 503 459 217 408 000,
 115 532 761 111 124 106 137 388 311 120 877 422 599 980 279 398 400,
 8 503 842 442 314 663 173 760 541 941 753 193 179 094 810 125 926 400,
 627 609 496 898 499 522 225 265 285 115 906 238 911 179 967 692 800 000,
 46 436 433 389 594 145 887 536 322 203 955 919 558 553 470 641 486 850 000,
 3 443 934 036 721 437 625 596 385 616 851 665 233 141 061 945 297 580 800 000,
 255 987 247 247 218 119 955 440 370 898 615 088 710 853 711 642 084 487 200 000,
 19 067 482 593 646 334 342 036 067 557 315 656 461 776 897 366 982 437 990 400 000,
 1 423 081 446 108 803 178 035 349 924 075 427 821 311 627 222 594 248 532 220 000 000,
 106 409 576 497 910 521 328 093 928 056 177 350 881 687 619 362 437 540 913 600 000 000,
 7 970 830 048 553 981 080 058 702 593 590 669 197 116 023 210 365 395 365 879 360 000 000,
 598 079 060 794 011 278 983 455 745 029 821 926 281 050 762 038 228 190 896 727 040 000 000,
 44 947 891 716 233 478 275 997 236 905 855 509 440 585 640 503 537 143 428 499 957 569 600 000,
 3 383 154 085 138 020 637 793 497 624 953 038 417 160 337 631 975 043 003 579 851 781 888 000 000}
```

Now we may numerically verify our REC.

```
In[•]:= Table[SeqNormalized /. {Seq → seq, α → n}, {n, 0, 2 MAX - RECNormalizedinSOrder}]

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Let us the generate a list of $r(n)$.

```
In[•]:= Bound = 5000;

SeqList = UnrollRecurrence[SeqNormalized, Seq[α], SeqListIni, Bound];

seq[n_] := SeqList[[n + 1]];
```

Guess a Minimal ODE for $R(z)$.

ODEGuessinTheta gives the ODE in Theorem 4.6! (To be displayed at the end of this notebook)

Order 9, Degree 24

```
In[•]:= ClearAll[Diff];
ODEGuess = GuessMinDE[Take[SeqList, 300], Diff[z]];
ODEGuessinD = NormalizeCoefficients[
   ToOrePolynomial[ODEGuess /. {Derivative[k_][Diff][z] → Der[z]^k} /. {Diff[z] → 1}]];

In[•]:= ODEGuessinTheta =
  NormalizeCoefficients[ChangeOreAlgebra[z ** ODEGuessinD, OreAlgebra[Euler[z]]]];

In[•]:= ODEGuessinThetaOrder = OrePolynomialDegree[ODEGuessinTheta, Euler[z]]

Out[•]= 9
```

In[∘]:=
```
ODEGuessinThetaDegree =
  Max[Exponent[OrePolynomialListCoefficients[ODEGuessinTheta], z]]
```
Out[∘]=
```
24
```

Get the REC from ODE and write it explicitly.

In[∘]:=
```
RECfromODEGuessinS = DFiniteDE2RE[{ODEGuessinD}, {z}, {α}][[1]];
```

In[∘]:=
```
RECfromODEGuessinSOrder = OrePolynomialDegree[RECfromODEGuessinS, S[α]]
```
Out[∘]=
```
24
```

In[∘]:=
```
ClearAll[Seq];
SeqfromODEGuess = ApplyOreOperator[RECfromODEGuessinS, Seq[α]];
```

In[∘]:=
```
SeqfromODEGuessList = UnrollRecurrence[
    SeqfromODEGuess, Seq[α], Take[SeqList, RECfromODEGuessinSOrder], 200];
```

Prove the minimal ODE for *R*(*z*).

In[∘]:=
```
RECCompare = DFinitePlus[{RECNormalizedinS}, {RECfromODEGuessinS}][[1]];
```

Compute the *largest* positive integral root of the leading coefficient in the recurrence RECCompare.

In[∘]:=
```
LeadCoeff = RECCompare[[1, 1, 1]];
LeadCoeffRoot = Solve[LeadCoeff == 0, α][[All, 1, 2]]
```
Out[∘]=
```
{-30, -30, -30, -30, -30, -59/2, -29, -29, -29, -29, -57/2, -28, … 244 …,
 Root[ … 1 … &, 239], Root[ … 1 … &, 240], Root[ … 1 … &, 241],
 Root[ … 1 … &, 242], Root[ … 1 … &, 243], Root[ … 1 … &, 244],
 Root[ … 1 … &, 245], Root[ … 1 … &, 246], Root[ … 1 … &, 247],
 Root[ … 1 … &, 248], Root[ … 1 … &, 249], Root[ … 1 … &, 250]}
```
large output | show less | show more | show all | set size limit...

There are no positive integral roots in our case.

In[∘]:=
```
Select[Select[LeadCoeffRoot, IntegerQ], # > 0 &]
```
Out[∘]=
```
{}
```

In[∘]:=
```
RECCompareOrder = OrePolynomialDegree[RECCompare, S[α]]
```
Out[∘]=
```
30
```

In[∘]:=
```
CheckNum = RECCompareOrder + 20;
Take[SeqList, CheckNum] - Take[SeqfromODEGuessList, CheckNum]
```
Out[∘]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Guess a Minimal REC for *r*(*n*).

SeqfromRECGuess gives the REC in Theorem 4.7! (To be displayed at the end of this notebook)

REC: Order 6

ODE: Order 33, Degree 6

```
In[•]:= RECGuess = GuessMinRE[Take[SeqList, 300], Seq[α]];
        RECGuessinS = NormalizeCoefficients[ToOrePolynomial[RECGuess /. {Seq[k_] → S[α]^(k-α)}]];

In[•]:= RECGuessinSOrder = OrePolynomialDegree[RECGuessinS, S[α]]

Out[•]= 6

In[•]:= ODEfromRECGuessinD =
          NormalizeCoefficients[DFiniteRE2DE[{RECGuessinS}, {α}, {z}][[1]]];

In[•]:= ODEfromRECGuessinTheta = NormalizeCoefficients[
           ChangeOreAlgebra[z ** ODEfromRECGuessinD, OreAlgebra[Euler[z]]]];

In[•]:= ODEfromRECGuessinThetaOrder = OrePolynomialDegree[ODEfromRECGuessinTheta, Euler[z]]

Out[•]= 33

In[•]:= ODEfromRECGuessinThetaDegree =
         Max[Exponent[OrePolynomialListCoefficients[ODEfromRECGuessinTheta], z]]

Out[•]= 6
```

We may also write this REC explicitly.

```
In[•]:= ClearAll[Seq];
        SeqfromRECGuess = ApplyOreOperator[RECGuessinS, Seq[α]];

In[•]:= SeqfromRECGuessList =
          UnrollRecurrence[SeqfromRECGuess, Seq[α], Take[SeqList, RECGuessinSOrder], 200];
```

Prove the minimal REC for *r*(*n*).

```
In[•]:= RECCompare = DFinitePlus[{RECNormalizedinS}, {RECGuessinS}][[1]];
```

Compute the *largest* positive integral root of the leading coefficient in the recurrence RECCompare.

```
In[•]:= LeadCoeff = RECCompare[[1, 1, 1]];
        LeadCoeffRoot = Solve[LeadCoeff == 0, α][[All, 1, 2]]

Out[•]= {-12, -12, -12, ⋯ 65 ⋯, Root[⋯ 1 ⋯ &, 51], Root[⋯ 1 ⋯ &, 52],
         Root[23 222 922 117 694 252 646 156 119 385 583 278 521 361 363 770 973 837 230 080 000 000 000 +
            284 117 715 670 835 864 313 213 787 362 394 727 279 530 132 862 347 765 388 697 600 000 000
             #1 + ⋯ 50 ⋯ + 893 341 938 205 210 059 214 453 053 849 600 #1^52 +
            3 038 208 948 104 686 221 062 150 553 600 #1^53 &, 53]}

        large output | show less | show more | show all | set size limit...
```

There are no positive integral roots in our case.

```
In[•]:= Select[Select[LeadCoeffRoot, IntegerQ], # > 0 &]

Out[•]= {}

In[•]:= RECCompareOrder = LeadingExponent[RECCompare][[1]]

Out[•]= 12

In[•]:= CheckNum = RECCompareOrder + 20;
        Take[SeqList, CheckNum] - Take[SeqfromRECGuessList, CheckNum]

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Compute the asymptotics for $r(n)$.

In[∘]:=
```
AsyList = Asymptotics[SeqfromRECGuess, Seq[α]];
N[AsyList]
```

Out[∘]= $\left\{\frac{(-432.)^{\alpha}}{\alpha^{5/2}}, \frac{(-48.)^{\alpha}}{\alpha^{5/2}}, \frac{(-5.33333)^{\alpha}}{\alpha^{5/2}}, \frac{16.^{\alpha}}{\alpha^{9/4}}, \frac{16.^{\alpha}}{\alpha^{7/4}}, \frac{80.^{\alpha}}{\alpha^{5/2}}\right\}$

In[∘]:=
```
Ind = Reverse[Table[Floor[Bound / i], {i, 1, 3}]]
Table[N[seq[Ind[[i]]] / (AsyList[[4]] /. {α → Ind[[i]]})], {i, 1, Length@Ind}]
Table[N[seq[Ind[[i]]] / (AsyList[[5]] /. {α → Ind[[i]]})], {i, 1, Length@Ind}]
Table[N[seq[Ind[[i]]] / (AsyList[[6]] /. {α → Ind[[i]]})], {i, 1, Length@Ind}]
```

Out[∘]= {3333, 5000, 10 000}

Out[∘]= $\{2.157784655879568\times10^{2327}, 2.971843676012373\times10^{3492}, 1.769474996617337\times10^{6987}\}$

Out[∘]= $\{3.737579539425117\times10^{2325}, 4.202821631869412\times10^{3490}, 1.769474996617337\times10^{6985}\}$

Out[∘]= {0.0352933, 0.0352977, 0.0353021}

Approximate the Polya number.

In[∘]:=
```
AtOne = N[Sum[seq[n] * (1 / (2^MM Binomial[NN, MM]))^n, {n, 0, Bound}], 11]
N[1 - 1/AtOne, 10]
```

Out[∘]= 1.0158559936

Out[∘]= 0.01560850527

Display the ODE in Theorem 4.6

In[∘]:=
```
ODEGuessinTheta
```

Out[∘]=
$$(1\,968\,300 - 14\,377\,372\,992\, z - 31\,378\,944\,803\,328\, z^2 - 587\,599\,727\,984\,640\, z^3 - 11\,393\,107\,020\,720\,046\,080\, z^4 - 7\,512\,914\,091\,413\,564\,817\,408\, z^5 + 299\,638\,067\,426\,947\,151\,953\,920\, z^6 + 195\,572\,469\,268\,564\,090\,225\,164\,288\, z^7 - 25\,066\,230\,988\,181\,914\,756\,830\,986\,240\, z^8 + 1\,466\,023\,585\,546\,150\,566\,663\,720\,796\,160\, z^9 + 71\,839\,838\,988\,731\,444\,762\,798\,769\,307\,648\, z^{10} - 8\,620\,981\,873\,487\,530\,449\,442\,157\,746\,978\,816\, z^{11} - 107\,877\,900\,379\,022\,416\,281\,433\,704\,771\,878\,912\, z^{12} + 6\,045\,203\,063\,427\,555\,738\,693\,218\,864\,495\,329\,280\, z^{13} - 27\,383\,749\,995\,592\,913\,844\,335\,383\,773\,613\,916\,160\, z^{14} + 44\,159\,405\,750\,235\,818\,360\,995\,501\,107\,081\,904\,128\, z^{15} + 13\,699\,073\,426\,625\,876\,523\,234\,327\,944\,587\,328\,356\,352\, z^{16} - 387\,340\,817\,532\,181\,412\,702\,477\,239\,142\,346\,601\,267\,200\, z^{17} - 93\,561\,082\,878\,589\,380\,479\,405\,717\,324\,153\,487\,360\,000\, z^{18} + 26\,199\,174\,990\,188\,349\,028\,511\,624\,137\,716\,063\,535\,104\,000\, z^{19} + 43\,846\,547\,777\,265\,123\,304\,897\,934\,342\,541\,583\,319\,040\,000\, z^{20} - 73\,654\,449\,615\,358\,974\,329\,157\,395\,854\,519\,173\,120\,000\,000\, z^{21} + 463\,163\,910\,329\,304\,284\,499\,157\,507\,080\,361\,869\,312\,000\,000\, z^{22} - 26\,729\,974\,870\,757\,227\,259\,400\,450\,682\,776\,453\,120\,000\,000\, z^{23} + 25\,483\,039\,017\,248\,114\,833\,274\,026\,825\,089\,024\,000\,000\,000\, z^{24})\,\theta_z^9 + (-9\,841\,500 + 91\,775\,477\,952\, z + 176\,504\,510\,301\,696\, z^2 - 60\,855\,583\,637\,790\,720\, z^3 + 137\,824\,643\,270\,780\,190\,720\, z^4 + 29\,196\,985\,244\,400\,911\,646\,720\, z^5 -$$

$1\,459\,547\,009\,100\,032\,948\,305\,920\,z^6 + 30\,525\,782\,594\,475\,535\,991\,046\,144\,z^7 +$
$36\,364\,502\,594\,318\,953\,136\,217\,128\,960\,z^8 - 11\,683\,073\,573\,344\,251\,270\,022\,813\,450\,240\,z^9 +$
$188\,342\,626\,264\,242\,759\,594\,195\,906\,723\,840\,z^{10} -$
$38\,493\,977\,475\,756\,415\,221\,342\,109\,479\,993\,344\,z^{11} +$
$197\,500\,136\,641\,585\,251\,203\,727\,542\,141\,845\,504\,z^{12} +$
$79\,339\,096\,438\,575\,233\,822\,624\,210\,434\,916\,352\,000\,z^{13} -$
$358\,839\,167\,901\,079\,462\,557\,845\,914\,072\,865\,832\,960\,z^{14} -$
$8\,071\,219\,003\,523\,395\,649\,517\,571\,342\,231\,142\,400\,000\,z^{15} +$
$6\,803\,000\,071\,934\,453\,973\,343\,268\,891\,476\,264\,747\,008\,z^{16} -$
$3\,815\,269\,798\,033\,428\,405\,284\,592\,607\,275\,218\,947\,276\,800\,z^{17} +$
$5\,724\,713\,912\,474\,141\,565\,314\,675\,653\,378\,943\,483\,904\,000\,z^{18} +$
$480\,538\,142\,467\,823\,411\,078\,212\,335\,942\,727\,498\,727\,424\,000\,z^{19} +$
$1\,070\,136\,609\,367\,513\,169\,695\,412\,587\,225\,689\,568\,051\,200\,000\,z^{20} -$
$1\,663\,432\,850\,480\,302\,478\,842\,623\,904\,156\,471\,001\,088\,000\,000\,z^{21} +$
$8\,359\,940\,894\,280\,588\,898\,973\,856\,579\,441\,955\,700\,736\,000\,000\,z^{22} -$
$468\,389\,851\,152\,613\,148\,630\,839\,940\,212\,630\,487\,040\,000\,000\,z^{23} +$
$407\,728\,624\,275\,969\,837\,332\,384\,429\,201\,424\,384\,000\,000\,000\,z^{24}\Big)\,\Theta_z^8 +$
$\Big(17\,222\,625 - 188\,109\,529\,956\,z - 450\,539\,864\,395\,776\,z^2 + 262\,908\,605\,083\,645\,440\,z^3 -$
$425\,793\,053\,888\,332\,259\,328\,z^4 - 47\,879\,449\,539\,860\,741\,750\,784\,z^5 +$
$19\,566\,005\,397\,726\,900\,761\,395\,200\,z^6 - 814\,444\,982\,834\,994\,376\,819\,605\,504\,z^7 +$
$74\,941\,704\,564\,121\,516\,865\,539\,276\,800\,z^8 - 6\,462\,799\,394\,578\,907\,339\,177\,655\,795\,712\,z^9 +$
$4\,202\,887\,839\,968\,581\,771\,721\,159\,573\,766\,144\,z^{10} -$
$156\,203\,798\,588\,367\,620\,630\,704\,585\,994\,928\,128\,z^{11} +$
$5\,671\,791\,513\,906\,639\,879\,948\,201\,111\,528\,144\,896\,z^{12} +$
$390\,474\,791\,519\,150\,913\,975\,998\,069\,242\,215\,792\,640\,z^{13} -$
$4\,473\,411\,603\,105\,987\,176\,029\,413\,018\,431\,926\,566\,912\,z^{14} -$
$89\,648\,235\,775\,403\,267\,396\,729\,942\,601\,723\,832\,958\,976\,z^{15} -$
$340\,815\,704\,642\,582\,411\,522\,200\,639\,822\,651\,881\,160\,704\,z^{16} -$
$16\,769\,525\,627\,605\,508\,461\,541\,721\,486\,157\,516\,741\,017\,600\,z^{17} +$
$39\,730\,598\,877\,359\,336\,156\,209\,541\,187\,847\,078\,281\,216\,000\,z^{18} +$
$4\,052\,905\,497\,254\,620\,526\,705\,033\,578\,997\,072\,888\,070\,144\,000\,z^{19} +$
$9\,918\,307\,911\,192\,702\,442\,375\,798\,488\,951\,038\,190\,551\,040\,000\,z^{20} -$
$14\,845\,027\,462\,578\,007\,694\,413\,631\,150\,601\,176\,875\,008\,000\,000\,z^{21} +$
$65\,889\,989\,953\,047\,210\,152\,655\,055\,449\,486\,438\,432\,768\,000\,000\,z^{22} -$
$3\,426\,880\,221\,784\,735\,083\,809\,689\,939\,817\,728\,573\,440\,000\,000\,z^{23} +$
$2\,858\,170\,577\,552\,599\,324\,112\,561\,161\,472\,311\,296\,000\,000\,000\,z^{24}\Big)\,\Theta_z^7 +$
$\Big(-12\,301\,875 + 147\,857\,370\,678\,z + 558\,785\,665\,638\,432\,z^2 - 348\,888\,960\,668\,305\,152\,z^3 +$
$623\,927\,792\,319\,268\,773\,888\,z^4 + 86\,547\,313\,967\,954\,563\,399\,680\,z^5 -$
$47\,520\,013\,366\,049\,481\,941\,188\,608\,z^6 + 8\,073\,642\,867\,629\,939\,237\,399\,298\,048\,z^7 -$
$481\,015\,401\,942\,302\,316\,411\,955\,970\,048\,z^8 - 83\,662\,110\,889\,859\,917\,447\,799\,209\,197\,568\,z^9 +$
$7\,372\,626\,647\,363\,540\,238\,695\,908\,528\,619\,520\,z^{10} -$
$517\,286\,141\,211\,413\,085\,726\,781\,671\,125\,024\,768\,z^{11} +$
$20\,299\,636\,115\,142\,546\,092\,262\,225\,115\,839\,725\,568\,z^{12} +$
$1\,032\,611\,462\,541\,549\,258\,786\,628\,905\,874\,868\,928\,512\,z^{13} -$
$23\,089\,987\,887\,622\,119\,469\,848\,577\,923\,864\,632\,229\,888\,z^{14} -$
$581\,255\,095\,048\,071\,795\,431\,779\,288\,334\,033\,045\,422\,080\,z^{15} -$
$2\,231\,423\,114\,980\,507\,246\,731\,626\,803\,096\,026\,643\,169\,280\,z^{16} -$
$52\,225\,166\,005\,254\,416\,359\,639\,117\,930\,743\,510\,073\,344\,000\,z^{17} +$
$64\,501\,775\,991\,653\,793\,038\,556\,666\,527\,909\,356\,240\,896\,000\,z^{18} +$
$19\,836\,063\,561\,165\,253\,941\,592\,255\,704\,184\,395\,738\,906\,624\,000\,z^{19} +$
$50\,367\,354\,002\,430\,652\,612\,982\,307\,804\,450\,170\,653\,900\,800\,000\,z^{20} -$
$72\,262\,673\,755\,836\,738\,546\,496\,170\,215\,886\,249\,000\,960\,000\,000\,z^{21} +$
$297\,661\,787\,964\,600\,264\,656\,433\,765\,601\,639\,969\,849\,344\,000\,000\,z^{22} -$
$13\,832\,482\,216\,433\,508\,202\,372\,696\,746\,522\,102\,988\,800\,000\,000\,z^{23} +$
$11\,526\,651\,016\,586\,499\,719\,897\,942\,619\,806\,760\,960\,000\,000\,000\,z^{24}\Big)\,\Theta_z^6 +$

$\left(2\,952\,450 - 36\,953\,052\,282\,z - 287\,925\,540\,888\,384\,z^2 + 240\,931\,070\,097\,037\,056\,z^3 - 289\,331\,557\,692\,340\,211\,712\,z^4 - 128\,387\,485\,397\,965\,538\,230\,272\,z^5 + 27\,573\,756\,410\,310\,410\,098\,704\,384\,z^6 - 1\,814\,822\,934\,375\,327\,328\,770\,195\,456\,z^7 + 98\,003\,249\,106\,798\,007\,207\,847\,788\,544\,z^8 - 37\,463\,350\,148\,731\,743\,364\,281\,121\,374\,208\,z^9 + 11\,803\,201\,206\,895\,787\,290\,523\,605\,760\,212\,992\,z^{10} - 931\,951\,868\,483\,519\,575\,665\,888\,059\,740\,651\,520\,z^{11} + 37\,270\,695\,716\,603\,566\,362\,564\,764\,251\,690\,369\,024\,z^{12} + 1\,693\,355\,628\,237\,010\,455\,481\,917\,104\,597\,514\,584\,064\,z^{13} - 74\,562\,930\,387\,006\,958\,127\,703\,563\,824\,328\,047\,853\,568\,z^{14} - 2\,207\,111\,793\,128\,944\,791\,191\,754\,382\,947\,658\,946\,838\,528\,z^{15} - 7\,366\,133\,899\,845\,447\,610\,553\,875\,349\,700\,012\,442\,386\,432\,z^{16} - 135\,724\,732\,404\,466\,580\,950\,581\,404\,009\,524\,691\,258\,572\,800\,z^{17} - 181\,585\,371\,776\,572\,020\,185\,148\,879\,468\,420\,390\,715\,392\,000\,z^{18} + 61\,259\,759\,653\,374\,654\,910\,414\,683\,422\,841\,093\,180\,358\,656\,000\,z^{19} + 158\,031\,105\,136\,778\,799\,601\,621\,847\,662\,535\,326\,732\,124\,160\,000\,z^{20} - 216\,009\,579\,954\,757\,701\,442\,160\,309\,718\,306\,313\,469\,952\,000\,000\,z^{21} + 849\,130\,590\,741\,023\,426\,723\,060\,816\,209\,756\,739\,862\,528\,000\,000\,z^{22} - 33\,985\,658\,167\,894\,423\,502\,801\,885\,737\,927\,342\,817\,280\,000\,000\,z^{23} + 29\,484\,628\,246\,538\,175\,143\,265\,003\,304\,780\,824\,576\,000\,000\,000\,z^{24}\right)\,\theta_z^5 +$

$\left(-136\,796\,850\,z + 7\,117\,846\,241\,760\,z^2 - 8\,844\,406\,827\,782\,400\,z^3 + 20\,673\,736\,810\,353\,008\,640\,z^4 + 12\,490\,222\,714\,507\,650\,170\,880\,z^5 - 12\,122\,002\,729\,261\,073\,154\,834\,432\,z^6 + 2\,882\,065\,176\,288\,698\,695\,601\,356\,800\,z^7 - 892\,798\,606\,426\,827\,006\,153\,137\,848\,320\,z^8 - 92\,995\,174\,917\,120\,951\,312\,035\,054\,878\,720\,z^9 + 17\,566\,600\,704\,846\,379\,365\,602\,311\,368\,867\,840\,z^{10} - 1\,106\,108\,983\,819\,645\,870\,965\,049\,203\,913\,392\,128\,z^{11} + 32\,780\,268\,668\,178\,750\,626\,003\,736\,845\,468\,303\,360\,z^{12} + 2\,110\,484\,779\,044\,380\,857\,170\,289\,665\,792\,708\,444\,160\,z^{13} - 151\,118\,551\,549\,948\,496\,465\,383\,948\,454\,238\,722\,457\,600\,z^{14} - 5\,208\,889\,177\,402\,096\,896\,569\,915\,682\,790\,363\,826\,749\,440\,z^{15} - 15\,214\,876\,436\,659\,767\,820\,660\,543\,018\,404\,039\,861\,731\,328\,z^{16} - 275\,047\,243\,698\,385\,265\,849\,237\,546\,494\,064\,761\,975\,603\,200\,z^{17} - 966\,664\,702\,875\,373\,589\,438\,035\,375\,364\,297\,579\,298\,816\,000\,z^{18} + 123\,152\,416\,705\,146\,488\,039\,473\,231\,196\,360\,524\,657\,852\,416\,000\,z^{19} + 320\,021\,187\,420\,662\,662\,460\,501\,139\,794\,039\,125\,573\,632\,000\,000\,z^{20} - 415\,625\,514\,130\,229\,218\,872\,683\,172\,052\,966\,607\,683\,584\,000\,000\,z^{21} + 1\,585\,673\,828\,397\,663\,133\,123\,332\,022\,196\,469\,208\,449\,024\,000\,000\,z^{22} - 52\,537\,320\,104\,709\,137\,746\,954\,097\,744\,194\,735\,964\,160\,000\,000\,z^{23} + 49\,626\,847\,003\,088\,039\,242\,732\,015\,341\,111\,607\,296\,000\,000\,000\,z^{24}\right)\,\theta_z^4 +$

$\left(-50\,191\,650\,z + 1\,738\,913\,583\,168\,z^2 - 3\,682\,056\,364\,704\,000\,z^3 + 11\,410\,666\,646\,947\,319\,808\,z^4 + 3\,861\,392\,978\,791\,762\,919\,424\,z^5 - 10\,019\,399\,490\,010\,425\,192\,873\,984\,z^6 + 2\,070\,503\,665\,419\,487\,435\,771\,871\,232\,z^7 - 938\,217\,822\,563\,635\,490\,605\,624\,197\,120\,z^8 - 87\,690\,228\,262\,864\,514\,350\,645\,361\,246\,208\,z^9 + 15\,716\,828\,962\,700\,965\,051\,167\,111\,020\,281\,856\,z^{10} - 835\,390\,587\,847\,491\,453\,520\,514\,214\,774\,439\,936\,z^{11} + 6\,217\,055\,792\,178\,871\,084\,099\,370\,009\,118\,113\,792\,z^{12} + 2\,261\,817\,830\,824\,757\,201\,413\,960\,797\,660\,968\,386\,560\,z^{13} - 198\,921\,399\,139\,980\,482\,757\,663\,766\,878\,256\,940\,187\,648\,z^{14} - 7\,820\,874\,309\,692\,886\,414\,163\,749\,280\,130\,712\,924\,585\,984\,z^{15} - 19\,966\,675\,134\,893\,089\,788\,479\,864\,838\,672\,568\,849\,268\,736\,z^{16} - 389\,109\,454\,617\,466\,345\,181\,082\,626\,107\,240\,683\,562\,598\,400\,z^{17} - 1\,833\,511\,986\,088\,921\,911\,263\,528\,204\,845\,572\,327\,211\,008\,000\,z^{18} + 160\,823\,135\,920\,101\,933\,029\,856\,888\,143\,866\,711\,083\,319\,296\,000\,z^{19} + 419\,500\,776\,084\,220\,530\,900\,484\,599\,900\,837\,509\,238\,620\,160\,000\,z^{20} - 517\,833\,585\,755\,647\,315\,897\,587\,665\,958\,962\,655\,657\,984\,000\,000\,z^{21} + 1\,937\,811\,509\,214\,205\,004\,760\,375\,531\,789\,832\,905\,818\,112\,000\,000\,z^{22} - 50\,656\,854\,326\,386\,655\,428\,015\,191\,380\,078\,006\,108\,160\,000\,000\,z^{23} + 54\,978\,816\,093\,356\,336\,026\,778\,587\,516\,605\,825\,024\,000\,000\,000\,z^{24}\right)\,\theta_z^3 +$

$\big(-5\,904\,900\,z + 153\,931\,767\,552\,z^2 - 1\,165\,603\,599\,249\,408\,z^3 + 4\,452\,725\,364\,540\,383\,232\,z^4 -$
$68\,100\,132\,399\,682\,682\,880\,z^5 - 6\,022\,639\,501\,601\,941\,259\,550\,720\,z^6 +$
$1\,416\,346\,779\,058\,053\,089\,914\,257\,408\,z^7 - 640\,104\,790\,763\,971\,096\,085\,771\,845\,632\,z^8 -$
$63\,985\,122\,201\,304\,857\,944\,332\,028\,608\,512\,z^9 + 9\,100\,063\,955\,033\,330\,138\,690\,098\,173\,050\,880\,z^{10} -$
$325\,020\,891\,478\,255\,709\,206\,452\,633\,600\,000\,000\,z^{11} -$
$15\,369\,973\,333\,180\,637\,978\,463\,636\,365\,185\,646\,592\,z^{12} +$
$1\,976\,820\,575\,315\,274\,068\,153\,581\,707\,743\,270\,535\,168\,z^{13} -$
$164\,845\,034\,045\,121\,253\,763\,793\,778\,737\,687\,142\,858\,752\,z^{14} -$
$7\,228\,405\,496\,467\,029\,747\,879\,519\,086\,728\,456\,773\,304\,320\,z^{15} -$
$16\,239\,487\,990\,047\,775\,727\,805\,633\,496\,882\,214\,149\,816\,320\,z^{16} -$
$350\,358\,271\,292\,264\,134\,491\,823\,082\,582\,765\,104\,791\,552\,000\,z^{17} -$
$1\,827\,945\,063\,176\,031\,872\,033\,100\,015\,820\,264\,710\,340\,608\,000\,z^{18} +$
$131\,473\,611\,752\,610\,706\,234\,343\,500\,650\,458\,530\,411\,708\,416\,000\,z^{19} +$
$343\,787\,530\,634\,088\,011\,151\,305\,145\,173\,989\,803\,845\,222\,400\,000\,z^{20} -$
$404\,288\,773\,803\,413\,717\,976\,988\,623\,723\,801\,914\,376\,192\,000\,000\,z^{21} +$
$1\,494\,111\,503\,773\,889\,636\,300\,556\,620\,388\,615\,529\,168\,896\,000\,000\,z^{22} -$
$28\,835\,289\,112\,340\,864\,117\,058\,495\,141\,988\,270\,080\,000\,000\,000\,z^{23} +$
$38\,666\,751\,190\,763\,482\,853\,658\,564\,366\,308\,474\,880\,000\,000\,000\,z^{24}\big)\,\theta_z^2 +$
$\big(-3\,779\,136\,000\,z^2 - 269\,104\,104\,907\,776\,z^3 + 1\,156\,924\,170\,186\,227\,712\,z^4 -$
$745\,981\,152\,037\,915\,852\,800\,z^5 - 2\,200\,561\,093\,127\,211\,319\,296\,000\,z^6 +$
$679\,764\,525\,023\,032\,711\,776\,829\,440\,z^7 - 247\,483\,767\,070\,577\,201\,125\,716\,393\,984\,z^8 -$
$28\,828\,824\,502\,077\,193\,882\,761\,934\,405\,632\,z^9 + 3\,009\,750\,779\,271\,795\,438\,756\,591\,682\,191\,360\,z^{10} -$
$24\,121\,851\,141\,176\,321\,998\,628\,141\,295\,206\,400\,z^{11} -$
$14\,041\,595\,223\,411\,212\,751\,132\,001\,452\,970\,475\,520\,z^{12} +$
$1\,099\,497\,577\,524\,424\,331\,870\,756\,199\,346\,194\,087\,936\,z^{13} -$
$78\,233\,392\,139\,468\,367\,220\,402\,010\,368\,141\,858\,701\,312\,z^{14} -$
$3\,741\,068\,067\,838\,249\,532\,222\,091\,407\,248\,458\,441\,031\,680\,z^{15} -$
$7\,479\,218\,518\,139\,921\,168\,057\,029\,800\,752\,304\,252\,518\,400\,z^{16} -$
$178\,554\,116\,967\,206\,659\,887\,262\,987\,102\,869\,415\,526\,400\,000\,z^{17} -$
$955\,633\,015\,178\,869\,945\,484\,182\,442\,489\,352\,479\,047\,680\,000\,z^{18} +$
$61\,064\,940\,090\,036\,835\,969\,974\,659\,001\,716\,745\,468\,641\,280\,000\,z^{19} +$
$160\,005\,180\,545\,999\,924\,367\,790\,335\,884\,887\,653\,875\,712\,000\,000\,z^{20} -$
$179\,882\,141\,683\,054\,785\,668\,996\,107\,776\,438\,105\,538\,560\,000\,000\,z^{21} +$
$659\,448\,853\,989\,575\,961\,502\,987\,599\,546\,877\,351\,034\,880\,000\,000\,z^{22} -$
$8\,425\,271\,771\,835\,012\,849\,481\,981\,181\,955\,145\,728\,000\,000\,000\,z^{23} +$
$15\,668\,087\,270\,761\,145\,604\,520\,827\,430\,738\,329\,600\,000\,000\,000\,z^{24}\big)\,\theta_z +$
$\big(-26\,643\,815\,792\,640\,z^3 + 143\,660\,616\,874\,721\,280\,z^4 - 223\,591\,081\,142\,491\,545\,600\,z^5 -$
$362\,256\,374\,063\,523\,535\,257\,600\,z^6 + 148\,966\,499\,505\,065\,275\,529\,625\,600\,z^7 -$
$41\,346\,406\,562\,321\,512\,194\,543\,452\,160\,z^8 - 5\,735\,331\,486\,845\,791\,568\,431\,184\,609\,280\,z^9 +$
$430\,952\,573\,711\,893\,752\,602\,017\,608\,499\,200\,z^{10} +$
$15\,365\,208\,973\,175\,735\,160\,733\,973\,662\,924\,800\,z^{11} -$
$3\,899\,218\,278\,931\,332\,512\,973\,370\,119\,998\,668\,800\,z^{12} +$
$268\,542\,927\,231\,239\,653\,274\,019\,036\,968\,245\,002\,240\,z^{13} -$
$16\,192\,706\,326\,137\,610\,914\,572\,827\,445\,643\,895\,111\,680\,z^{14} -$
$827\,628\,100\,816\,628\,174\,031\,976\,824\,998\,601\,110\,323\,200\,z^{15} -$
$1\,492\,168\,620\,825\,463\,185\,266\,901\,480\,947\,935\,346\,688\,000\,z^{16} -$
$39\,029\,649\,238\,703\,972\,266\,689\,359\,525\,936\,784\,998\,400\,000\,z^{17} -$
$207\,256\,173\,204\,460\,228\,197\,474\,900\,138\,380\,387\,942\,400\,000\,z^{18} +$
$12\,284\,360\,408\,950\,237\,248\,944\,612\,553\,628\,906\,527\,129\,600\,000\,z^{19} +$
$32\,252\,946\,091\,064\,139\,421\,946\,313\,669\,223\,309\,639\,680\,000\,000\,z^{20} -$
$34\,803\,590\,327\,109\,990\,774\,681\,094\,113\,253\,864\,243\,200\,000\,000\,z^{21} +$
$126\,937\,046\,777\,747\,284\,444\,281\,895\,736\,867\,133\,849\,600\,000\,000\,z^{22} -$
$847\,913\,178\,135\,460\,876\,352\,019\,117\,543\,260\,160\,000\,000\,000\,z^{23} +$
$2\,787\,207\,392\,511\,512\,559\,889\,346\,683\,994\,112\,000\,000\,000\,000\,z^{24}\big)$

Display the REC in Theorem 4.7

In[∘]:=
```
Collect[Expand[-SeqfromRECGuess], Seq[_]]
```

Out[∘]=

$\big(2\,364\,822\,061\,925\,891\,270\,067\,722\,649\,600\,000 + 24\,311\,763\,241\,480\,737\,290\,507\,853\,496\,320\,000\,\alpha +$
$118\,884\,714\,388\,336\,585\,062\,289\,753\,767\,936\,000\,\alpha^{2} +$
$368\,251\,136\,151\,853\,255\,846\,369\,719\,798\,988\,800\,\alpha^{3} +$
$811\,793\,640\,582\,985\,414\,140\,746\,797\,028\,474\,880\,\alpha^{4} +$
$1\,356\,499\,120\,040\,750\,577\,583\,138\,444\,526\,223\,360\,\alpha^{5} +$
$1\,786\,835\,040\,377\,781\,128\,110\,811\,754\,937\,712\,640\,\alpha^{6} +$
$1\,904\,958\,007\,246\,824\,509\,445\,186\,467\,125\,002\,240\,\alpha^{7} +$
$1\,674\,545\,402\,297\,600\,373\,785\,511\,713\,251\,000\,320\,\alpha^{8} +$
$1\,230\,194\,808\,706\,317\,371\,163\,067\,050\,208\,788\,480\,\alpha^{9} +$
$762\,791\,807\,513\,049\,677\,466\,384\,009\,532\,538\,880\,\alpha^{10} +$
$402\,079\,430\,499\,218\,110\,643\,393\,128\,200\,929\,280\,\alpha^{11} +$
$181\,085\,303\,893\,806\,582\,831\,390\,648\,576\,245\,760\,\alpha^{12} +$
$69\,909\,566\,044\,762\,687\,837\,271\,137\,604\,075\,520\,\alpha^{13} +$
$23\,174\,037\,389\,797\,607\,720\,091\,614\,796\,840\,960\,\alpha^{14} +$
$6\,597\,237\,647\,955\,223\,324\,018\,009\,760\,071\,680\,\alpha^{15} +$
$1\,610\,851\,715\,462\,724\,269\,782\,004\,410\,613\,760\,\alpha^{16} +$
$336\,382\,193\,033\,012\,242\,367\,855\,858\,810\,880\,\alpha^{17} +$
$59\,795\,770\,083\,083\,316\,221\,336\,805\,703\,680\,\alpha^{18} + 8\,987\,061\,025\,545\,721\,077\,834\,511\,810\,560\,\alpha^{19} +$
$1\,131\,237\,375\,988\,193\,565\,613\,353\,861\,120\,\alpha^{20} + 117\,704\,523\,870\,056\,936\,584\,154\,972\,160\,\alpha^{21} +$
$9\,941\,030\,662\,497\,120\,749\,554\,237\,440\,\alpha^{22} + 664\,040\,244\,922\,741\,425\,721\,835\,520\,\alpha^{23} +$
$33\,746\,986\,442\,943\,554\,031\,452\,160\,\alpha^{24} + 1\,225\,566\,587\,608\,656\,091\,545\,600\,\alpha^{25} +$
$28\,320\,365\,528\,012\,449\,382\,400\,\alpha^{26} + 312\,808\,771\,118\,086\,225\,920\,\alpha^{27}\big)$ Seq[α] +

$\big(880\,540\,948\,213\,763\,261\,498\,004\,602\,880\,000 + 8\,086\,612\,414\,279\,581\,582\,690\,097\,299\,456\,000\,\alpha +$
$35\,535\,843\,625\,080\,580\,938\,628\,852\,403\,404\,800\,\alpha^{2} +$
$99\,482\,199\,073\,846\,865\,130\,149\,987\,053\,731\,840\,\alpha^{3} +$
$199\,278\,215\,238\,194\,877\,084\,174\,219\,759\,058\,944\,\alpha^{4} +$
$304\,147\,288\,569\,704\,121\,767\,283\,668\,058\,636\,288\,\alpha^{5} +$
$367\,726\,422\,460\,034\,552\,713\,877\,456\,306\,307\,072\,\alpha^{6} +$
$361\,508\,986\,147\,801\,089\,153\,130\,211\,095\,805\,952\,\alpha^{7} +$
$294\,331\,319\,744\,750\,632\,422\,172\,167\,712\,997\,376\,\alpha^{8} +$
$201\,108\,607\,972\,501\,732\,293\,906\,606\,562\,934\,784\,\alpha^{9} +$
$116\,437\,788\,942\,848\,727\,536\,075\,769\,222\,856\,704\,\alpha^{10} +$
$57\,524\,299\,296\,878\,619\,402\,424\,939\,339\,382\,784\,\alpha^{11} +$
$24\,367\,165\,878\,769\,872\,656\,509\,536\,747\,061\,248\,\alpha^{12} +$
$8\,877\,402\,295\,660\,764\,714\,512\,245\,808\,234\,496\,\alpha^{13} +$
$2\,785\,748\,984\,068\,408\,698\,625\,918\,477\,467\,648\,\alpha^{14} +$
$752\,972\,653\,647\,501\,430\,958\,086\,738\,673\,664\,\alpha^{15} +$
$175\,049\,743\,314\,674\,169\,771\,167\,299\,534\,848\,\alpha^{16} + 34\,895\,534\,864\,837\,208\,484\,258\,292\,957\,184\,\alpha^{17} +$
$5\,936\,277\,532\,573\,962\,980\,718\,997\,929\,984\,\alpha^{18} + 855\,818\,515\,821\,739\,179\,539\,429\,326\,848\,\alpha^{19} +$
$103\,560\,073\,600\,267\,246\,364\,541\,321\,216\,\alpha^{20} + 10\,380\,185\,487\,431\,012\,018\,005\,475\,328\,\alpha^{21} +$
$846\,180\,664\,706\,397\,472\,693\,420\,032\,\alpha^{22} + 54\,656\,640\,176\,185\,180\,963\,209\,216\,\alpha^{23} +$
$2\,690\,612\,916\,385\,314\,156\,576\,768\,\alpha^{24} + 94\,804\,345\,329\,795\,433\,758\,720\,\alpha^{25} +$
$2\,128\,785\,749\,082\,227\,343\,360\,\alpha^{26} + 22\,881\,382\,331\,785\,936\,896\,\alpha^{27}\big)$ Seq[1 + α] +

$\big(-664\,078\,540\,666\,702\,251\,488\,371\,015\,680\,000 - 5\,805\,956\,958\,011\,506\,960\,041\,778\,348\,032\,000\,\alpha -$
$24\,298\,272\,789\,380\,152\,495\,188\,221\,126\,246\,400\,\alpha^{2} -$
$64\,810\,405\,629\,301\,547\,428\,216\,819\,254\,558\,720\,\alpha^{3} -$
$123\,755\,374\,367\,469\,269\,296\,809\,845\,353\,611\,264\,\alpha^{4} -$
$180\,149\,375\,502\,996\,189\,202\,275\,648\,542\,982\,144\,\alpha^{5} -$
$207\,865\,771\,244\,125\,682\,287\,781\,841\,861\,722\,112\,\alpha^{6} -$
$195\,153\,222\,041\,523\,657\,876\,484\,723\,267\,989\,504\,\alpha^{7} -$
$151\,846\,270\,858\,495\,120\,363\,896\,477\,860\,167\,680\,\alpha^{8} -$
$99\,230\,231\,828\,276\,421\,932\,960\,434\,682\,314\,752\,\alpha^{9} -$

$54\,993\,115\,047\,787\,497\,911\,079\,580\,675\,899\,392\,\alpha^{10}-$
$26\,028\,017\,908\,489\,825\,928\,212\,462\,245\,453\,824\,\alpha^{11}-$
$10\,572\,113\,416\,646\,586\,933\,511\,582\,698\,766\,336\,\alpha^{12}-$
$3\,696\,722\,231\,163\,815\,760\,173\,082\,026\,344\,448\,\alpha^{13}-$
$1\,114\,468\,173\,061\,041\,282\,670\,805\,399\,093\,248\,\alpha^{14}-$
$289\,688\,969\,845\,746\,113\,335\,461\,572\,931\,584\,\alpha^{15}-$
$64\,831\,091\,647\,102\,802\,811\,533\,842\,055\,168\,\alpha^{16}-12\,454\,053\,009\,083\,005\,771\,527\,566\,163\,968\,\alpha^{17}-$
$2\,043\,760\,292\,966\,696\,499\,523\,264\,184\,320\,\alpha^{18}-284\,532\,912\,366\,921\,324\,027\,166\,588\,928\,\alpha^{19}-$
$33\,284\,416\,956\,384\,385\,896\,458\,223\,616\,\alpha^{20}-3\,228\,606\,478\,351\,534\,833\,828\,626\,432\,\alpha^{21}-$
$254\,974\,947\,491\,313\,890\,128\,560\,128\,\alpha^{22}-15\,972\,126\,457\,377\,261\,067\,698\,176\,\alpha^{23}-$
$763\,333\,007\,662\,980\,725\,211\,136\,\alpha^{24}-26\,138\,887\,552\,462\,651\,129\,856\,\alpha^{25}-$
$570\,997\,443\,951\,748\,710\,400\,\alpha^{26}-5\,976\,795\,675\,008\,958\,464\,\alpha^{27}\big)\ \mathrm{Seq}[2+\alpha]+$
$\big(36\,337\,840\,931\,616\,555\,318\,702\,833\,664\,000+310\,343\,693\,247\,202\,072\,877\,171\,431\,833\,600\,\alpha+$
$1\,268\,062\,726\,217\,635\,641\,408\,454\,051\,430\,400\,\alpha^{2}+$
$3\,300\,521\,955\,790\,071\,740\,463\,976\,232\,263\,680\,\alpha^{3}+6\,146\,984\,578\,367\,464\,065\,862\,054\,879\,242\,240$
$\alpha^{4}+8\,723\,512\,529\,514\,925\,026\,222\,139\,080\,468\,480\,\alpha^{5}+$
$9\,808\,817\,646\,565\,897\,068\,529\,809\,213\,239\,808\,\alpha^{6}+8\,970\,447\,157\,798\,999\,809\,214\,350\,039\,412\,224$
$\alpha^{7}+6\,796\,618\,106\,855\,403\,262\,931\,535\,421\,469\,184\,\alpha^{8}+$
$4\,323\,600\,610\,674\,086\,572\,350\,145\,316\,732\,416\,\alpha^{9}+2\,331\,860\,127\,398\,843\,166\,087\,931\,718\,971\,904$
$\alpha^{10}+1\,073\,804\,990\,271\,736\,796\,663\,841\,511\,156\,224\,\alpha^{11}+$
$424\,279\,297\,446\,148\,516\,898\,147\,199\,947\,264\,\alpha^{12}+144\,293\,344\,557\,135\,741\,340\,883\,292\,465\,664\,\alpha^{13}+$
$42\,304\,696\,119\,152\,808\,149\,756\,544\,291\,840\,\alpha^{14}+10\,693\,366\,157\,119\,575\,923\,154\,101\,714\,944\,\alpha^{15}+$
$2\,327\,102\,570\,668\,214\,059\,453\,238\,664\,192\,\alpha^{16}+434\,708\,874\,971\,795\,823\,099\,840\,116\,736\,\alpha^{17}+$
$69\,373\,988\,097\,051\,870\,247\,906\,934\,784\,\alpha^{18}+9\,393\,304\,762\,567\,159\,143\,035\,764\,736\,\alpha^{19}+$
$1\,068\,815\,757\,774\,279\,757\,481\,902\,080\,\alpha^{20}+100\,861\,570\,825\,855\,881\,262\,923\,776\,\alpha^{21}+$
$7\,750\,770\,733\,439\,394\,600\,976\,384\,\alpha^{22}+472\,551\,963\,878\,997\,639\,561\,216\,\alpha^{23}+$
$21\,986\,541\,883\,647\,884\,001\,280\,\alpha^{24}+733\,188\,729\,988\,561\,502\,208\,\alpha^{25}+$
$15\,602\,375\,112\,618\,147\,840\,\alpha^{26}+159\,149\,910\,074\,064\,896\,\alpha^{27}\big)\ \mathrm{Seq}[3+\alpha]+$
$\big(1\,737\,772\,868\,400\,007\,324\,872\,130\,560\,000+14\,528\,609\,204\,414\,291\,845\,066\,255\,564\,800\,\alpha+$
$58\,083\,087\,258\,852\,534\,411\,685\,975\,019\,520\,\alpha^{2}+147\,846\,850\,915\,658\,722\,383\,612\,355\,430\,400\,\alpha^{3}+$
$269\,164\,023\,324\,400\,460\,962\,054\,275\,740\,928\,\alpha^{4}+373\,240\,816\,513\,597\,979\,905\,593\,440\,661\,888\,\alpha^{5}+$
$409\,908\,879\,949\,766\,514\,326\,399\,060\,864\,064\,\alpha^{6}+366\,016\,393\,873\,249\,701\,940\,597\,734\,061\,344\,\alpha^{7}+$
$270\,676\,671\,846\,416\,971\,917\,873\,052\,917\,920\,\alpha^{8}+168\,013\,318\,310\,785\,666\,403\,759\,927\,887\,584\,\alpha^{9}+$
$88\,393\,926\,598\,940\,439\,065\,183\,725\,045\,600\,\alpha^{10}+39\,697\,363\,634\,496\,672\,642\,069\,844\,386\,912\,\alpha^{11}+$
$15\,293\,672\,611\,896\,263\,618\,803\,193\,519\,136\,\alpha^{12}+5\,070\,491\,874\,452\,377\,148\,797\,920\,831\,072\,\alpha^{13}+$
$1\,449\,002\,022\,519\,967\,409\,403\,051\,116\,512\,\alpha^{14}+356\,957\,682\,436\,813\,381\,749\,659\,746\,304\,\alpha^{15}+$
$75\,700\,244\,148\,872\,939\,301\,421\,992\,640\,\alpha^{16}+13\,779\,371\,789\,456\,905\,170\,877\,563\,840\,\alpha^{17}+$
$2\,142\,685\,081\,818\,193\,152\,012\,367\,872\,\alpha^{18}+282\,685\,926\,147\,777\,894\,282\,083\,328\,\alpha^{19}+$
$31\,341\,335\,886\,140\,485\,043\,322\,880\,\alpha^{20}+2\,881\,942\,426\,887\,984\,021\,438\,464\,\alpha^{21}+$
$215\,812\,414\,752\,103\,173\,455\,872\,\alpha^{22}+12\,823\,036\,513\,484\,289\,343\,488\,\alpha^{23}+$
$581\,508\,878\,853\,457\,575\,936\,\alpha^{24}+18\,903\,053\,117\,719\,314\,432\,\alpha^{25}+$
$392\,186\,219\,850\,629\,120\,\alpha^{26}+3\,900\,964\,176\,134\,144\,\alpha^{27}\big)\ \mathrm{Seq}[4+\alpha]+$
$\big(-36\,446\,102\,109\,669\,030\,849\,285\,120\,000-301\,794\,930\,778\,773\,719\,063\,321\,856\,000\,\alpha-$
$1\,194\,401\,836\,156\,084\,887\,609\,064\,224\,000\,\alpha^{2}-3\,008\,156\,975\,709\,477\,795\,289\,491\,275\,520\,\alpha^{3}-$
$5\,415\,770\,546\,395\,539\,670\,222\,530\,489\,360\,\alpha^{4}-7\,422\,453\,554\,874\,065\,600\,190\,474\,289\,032\,\alpha^{5}-$
$8\,052\,206\,383\,842\,449\,223\,124\,682\,104\,644\,\alpha^{6}-7\,098\,162\,826\,794\,167\,361\,280\,152\,144\,294\,\alpha^{7}-$
$5\,179\,144\,111\,408\,801\,590\,076\,035\,892\,950\,\alpha^{8}-3\,169\,950\,795\,733\,038\,711\,522\,140\,215\,280\,\alpha^{9}-$
$1\,643\,499\,248\,947\,095\,475\,104\,215\,404\,004\,\alpha^{10}-726\,910\,788\,718\,026\,537\,302\,273\,862\,144\,\alpha^{11}-$
$275\,635\,972\,025\,251\,416\,199\,969\,761\,656\,\alpha^{12}-89\,889\,728\,147\,001\,421\,773\,544\,625\,132\,\alpha^{13}-$
$25\,251\,994\,806\,501\,150\,584\,061\,125\,784\,\alpha^{14}-6\,111\,409\,098\,652\,595\,993\,659\,452\,026\,\alpha^{15}-$
$1\,272\,483\,225\,563\,071\,816\,917\,699\,490\,\alpha^{16}-227\,273\,250\,419\,552\,627\,170\,585\,084\,\alpha^{17}-$
$34\,655\,941\,701\,831\,856\,557\,922\,624\,\alpha^{18}-4\,480\,880\,404\,407\,427\,210\,024\,320\,\alpha^{19}-$
$486\,585\,842\,769\,876\,461\,484\,032\,\alpha^{20}-43\,798\,304\,089\,562\,788\,663\,296\,\alpha^{21}-$
$3\,208\,710\,131\,027\,557\,023\,744\,\alpha^{22}-186\,416\,522\,833\,559\,945\,216\,\alpha^{23}-8\,261\,380\,192\,874\,790\,912\,\alpha^{24}-$
$262\,301\,388\,296\,421\,376\,\alpha^{25}-5\,312\,632\,953\,241\,600\,\alpha^{26}-51\,561\,082\,388\,480\,\alpha^{27}\big)\ \mathrm{Seq}[5+\alpha]+$

$$(-154\,404\,486\,709\,237\,819\,219\,968\,000 - 1\,265\,327\,918\,255\,018\,927\,110\,348\,800\,\alpha - 4\,953\,641\,658\,930\,095\,511\,385\,751\,040\,\alpha^{2} - 12\,335\,446\,851\,783\,544\,166\,937\,390\,720\,\alpha^{3} - 21\,947\,702\,123\,383\,074\,616\,990\,244\,544\,\alpha^{4} - 29\,712\,684\,443\,300\,038\,100\,072\,561\,760\,\alpha^{5} - 31\,824\,626\,177\,807\,101\,870\,129\,360\,368\,\alpha^{6} - 27\,684\,339\,638\,906\,598\,652\,692\,786\,888\,\alpha^{7} - 19\,923\,668\,408\,873\,674\,929\,361\,243\,572\,\alpha^{8} - 12\,021\,754\,897\,932\,453\,908\,473\,126\,194\,\alpha^{9} - 6\,141\,402\,912\,303\,808\,338\,721\,284\,327\,\alpha^{10} - 2\,675\,090\,519\,652\,464\,763\,702\,625\,995\,\alpha^{11} - 998\,451\,712\,547\,824\,111\,144\,656\,513\,\alpha^{12} - 320\,337\,381\,856\,256\,276\,567\,115\,789\,\alpha^{13} - 88\,485\,146\,094\,830\,787\,771\,471\,525\,\alpha^{14} - 21\,045\,641\,782\,461\,353\,200\,898\,049\,\alpha^{15} - 4\,304\,140\,182\,149\,530\,399\,276\,227\,\alpha^{16} - 754\,678\,659\,252\,915\,954\,749\,073\,\alpha^{17} - 112\,910\,766\,050\,133\,819\,763\,020\,\alpha^{18} - 14\,316\,213\,223\,182\,938\,203\,068\,\alpha^{19} - 1\,523\,679\,350\,645\,560\,062\,336\,\alpha^{20} - 134\,345\,128\,624\,663\,841\,280\,\alpha^{21} - 9\,635\,762\,018\,738\,626\,560\,\alpha^{22} - 547\,760\,583\,383\,666\,688\,\alpha^{23} - 23\,739\,371\,943\,886\,848\,\alpha^{24} - 736\,693\,272\,182\,784\,\alpha^{25} - 14\,575\,541\,944\,320\,\alpha^{26} - 138\,110\,042\,112\,\alpha^{27})\ \mathrm{Seq}[6+\alpha]$$