Multi-headed Lattice Green Function (N = 4, M = 3)

```
In[•]:= NN = 4;
MM = 3;
```

Recall some basic definitions in the paper:

$$P_{M,N}(z) := \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{1}{1 - \frac{z}{\binom{N}{M}} \sigma_M(\cos\theta_1, \ldots, \cos\theta_N)} d\theta_1 \ldots d\theta_N$$

$$R_{M,N}(z) := P_{M,N}\left(2^M \binom{N}{M} z\right) \text{ and } R_{M,N}(z) = \sum_{n\geq 0} r_{M,N}(n)\, z^n$$

Also, for M odd or $M = N$, we always have $r(2n+1) = 0$. Hence, define
$\tilde{r}_{M,N}(n) := r_{M,N}(2n)$ and $\tilde{R}_{M,N}(z) := \sum_{n\geq 0} \tilde{r}_{M,N}(n)\, z^n = \sum_{n\geq 0} r_{M,N}(2n)\, z^n$

Our goal is to find:

Case 1. M even and $M \neq N$:

- recurrences (REC) for $r(n)$ or differential equations (ODE) for $R(z)$.

Case 2. M odd or $M = N$:

- recurrences (REC) for $\tilde{r}(n)$ or differential equations (ODE) for $\tilde{R}(z)$.

Command: UnrollRecurrence

Generate a sequence from recurrence & initial values (Koutschan's implementation).

```
In[•]:= (* given a recurrence rec in f[n], compute the values {f[0],f[1],...,f[bound]}
 where inits are the initial values
 {f[0],...,f[d-1]} with d being the order of the recurrence *)
Clear[UnrollRecurrence];
UnrollRecurrence[rec1_, f_[n_], inits_, bound_] :=
  Module[{i, x, vals = inits, rec = rec1},
   If[Head[rec] =!= Equal, rec = (rec == 0)];
   rec = rec /. n → n - Max[Cases[rec, f[n + a_.] :→ a, Infinity]];
   Do[
    AppendTo[vals, Solve[rec /. n → i /. f[i] → x /. f[a_] :→ vals[[a + 1]], x][[1, 1, 2]]];
    , {i, Length[inits], bound}];
   Return[vals];
  ];
```

Load RISC packages.

```
In[∘]:= << RISC`HolonomicFunctions`
        << RISC`Asymptotics`
        << RISC`Guess`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
--> Type  ?HolonomicFunctions  for help.
```

Asymptotics Package version 0.3
written by Manuel Kauers
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

Package GeneratingFunctions version 0.9 written by Christian Mallinger
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

Guess Package version 0.52
written by Manuel Kauers
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

Apply creative telescoping to $R(z/2^M)$.

```
In[∘]:= ClearAll[x1, x2, x3, x4, z, w, α, β];

In[∘]:= SymmetricPolynomial[3, {x1, x2, x3, x4}]
Out[∘]= x1 x2 x3 + x1 x2 x4 + x1 x3 x4 + x2 x3 x4

In[∘]:= integrand = 1 / ((1 - z (x1 x2 x3 + x1 x2 x4 + x1 x3 x4 + x2 x3 x4))
          Sqrt[1 - x1^2] Sqrt[1 - x2^2] Sqrt[1 - x3^2] Sqrt[1 - x4^2]);

In[∘]:= Timing[ann0 = Annihilator[integrand, {Der[x1], Der[x2], Der[x3], Der[x4], Der[z]}];]
Out[∘]= {0.03125, Null}

In[∘]:= Timing[ann1 = FindCreativeTelescoping[ann0, Der[x1]][[1]];]
Out[∘]= {0.8125, Null}

In[∘]:= Timing[ann2 = FindCreativeTelescoping[ann1, Der[x2]][[1]];]
Out[∘]= {3.89063, Null}

In[∘]:= Timing[ann3 = FindCreativeTelescoping[ann2, Der[x3]][[1]];]
```

Out[•]= {140.656, Null}

In[•]:=
```
Timing[ann4 = FindCreativeTelescoping[ann3, Der[x4]][[1]];]
```

Out[•]= {2214.66, Null}

Alternatively, you may import the value of ann4 from an external file.

In[•]:=
```
ann4 = ToExpression[Import[NotebookDirectory[] <> "Data-N4M3-Integral.txt"]];
```

ann4 gives an ODE for $R(z/2^M)$.

In[•]:=
```
ODEDiv2 = ann4[[1]];
```

Compute the ODE for $R(z)$.

ODEinD - in terms of the derivation operator D

ODEinTheta - in terms of the derivation operator θ - **Order 8, Degree 32 (Refer to Table 1)**

In[•]:=
```
ODETemp = NormalizeCoefficients[
   DFiniteSubstitute[{ODEDiv2}, {z → w * 2^MM}, Algebra → OreAlgebra[Der[w]]][[1]]];
```

In[•]:=
```
ODEinD = NormalizeCoefficients[
   DFiniteSubstitute[{ODETemp}, {w → z}, Algebra → OreAlgebra[Der[z]]][[1]]];
```

In[•]:=
```
ODEinTheta = NormalizeCoefficients[ChangeOreAlgebra[z ** ODEinD, OreAlgebra[Euler[z]]]];
```

In[•]:=
```
ODEinThetaOrder = OrePolynomialDegree[ODEinTheta, Euler[z]]
```

Out[•]= 8

In[•]:=
```
ODEinThetaDegree = Max[Exponent[OrePolynomialListCoefficients[ODEinTheta], z]]
```

Out[•]= 32

Since $M = 3$ is odd, we move on to the ODE for $\tilde{R}(z) = R(z^{1/2})$.

ODENormalizedinTheta gives the ODE in Theorem 4.2! (To be displayed at the end of this notebook)

Order 8, Degree 16

In[•]:=
```
ODENormalizedinD = NormalizeCoefficients[
   DFiniteSubstitute[{ODEinD}, {z → w^(1/2)}, Algebra → OreAlgebra[Der[w]]][[1]]];
```

In[•]:=
```
ODENormalizedinTheta =
  NormalizeCoefficients[ChangeOreAlgebra[w ** ODENormalizedinD, OreAlgebra[Euler[w]]]];
```

In[•]:=
```
ODENormalizedinThetaOrder = OrePolynomialDegree[ODENormalizedinTheta, Euler[w]]
```

Out[•]= 8

In[•]:=
```
ODENormalizedinThetaDegree =
 Max[Exponent[OrePolynomialListCoefficients[ODENormalizedinTheta], w]]
```

Out[•]= 16

Get the REC for $\tilde{r}(n)$.

Order 16

```
In[•]:= RECNormalizedinS = NormalizeCoefficients[DFiniteDE2RE[{ODENormalizedinD}, {w}, {α}][[1]]];

In[•]:= RecNormalizedinSOrder = OrePolynomialDegree[RECNormalizedinS, S[α]]

Out[•]= 16
```

We may also write this REC explicitly.

```
In[•]:= ClearAll[Seq];
        SeqNormalized = ApplyOreOperator[RECNormalizedinS, Seq[α]];
```

The initial values of $\tilde{r}(n)$ are also produced by the ODE for $\tilde{R}(z)$.

```
In[•]:= MAX = 20;
        ClearAll[a];

        SeriesIni = ApplyOreOperator[ODENormalizedinTheta, Sum[a[n] w^n, {n, 0, MAX}]];

        SeriesIniSol = Solve[Join[Table[Coefficient[SeriesIni, w, i] == 0, {i, 1, MAX}], {a[0] == 1}],
          Table[a[i], {i, 0, MAX}]]

        SeqListIni = Table[SeriesIniSol[[1, k, 2]], {k, 1, Length@SeriesIniSol[[1]]}]

        seq[n_] := SeqListIni[[n + 1]];

Out[•]= {{a[0] → 1, a[1] → 32, a[2] → 6048, a[3] → 2 451 200, a[4] → 1 391 236 000, a[5] → 921 422 380 032,
          a[6] → 663 895 856 219 904, a[7] → 505 041 413 866 868 736, a[8] → 399 445 932 990 555 902 880,
          a[9] → 325 440 143 503 901 735 429 120, a[10] → 271 445 584 301 606 582 663 031 808,
          a[11] → 230 773 066 339 125 955 854 130 661 376, a[12] → 199 326 200 240 673 646 611 787 771 995 904,
          a[13] → 174 478 237 021 099 598 812 491 315 604 889 600,
          a[14] → 154 480 035 620 813 053 446 642 174 412 128 768 000,
          a[15] → 138 129 336 609 134 098 952 004 475 839 318 761 472 000,
          a[16] → 124 577 089 053 969 968 356 059 653 140 361 638 344 938 400,
          a[17] → 113 209 463 052 287 193 655 237 025 876 331 530 870 707 737 600,
          a[18] → 103 573 496 015 054 055 969 039 980 718 499 533 706 000 571 520 000,
          a[19] → 95 328 837 240 197 678 160 114 853 748 204 677 385 026 223 109 120 000,
          a[20] → 88 215 610 025 056 975 283 519 690 346 309 846 200 279 286 296 474 496 000}}

Out[•]= {1, 32, 6048, 2 451 200, 1 391 236 000, 921 422 380 032, 663 895 856 219 904, 505 041 413 866 868 736,
         399 445 932 990 555 902 880, 325 440 143 503 901 735 429 120, 271 445 584 301 606 582 663 031 808,
         230 773 066 339 125 955 854 130 661 376, 199 326 200 240 673 646 611 787 771 995 904,
         174 478 237 021 099 598 812 491 315 604 889 600, 154 480 035 620 813 053 446 642 174 412 128 768 000,
         138 129 336 609 134 098 952 004 475 839 318 761 472 000,
         124 577 089 053 969 968 356 059 653 140 361 638 344 938 400,
         113 209 463 052 287 193 655 237 025 876 331 530 870 707 737 600,
         103 573 496 015 054 055 969 039 980 718 499 533 706 000 571 520 000,
         95 328 837 240 197 678 160 114 853 748 204 677 385 026 223 109 120 000,
         88 215 610 025 056 975 283 519 690 346 309 846 200 279 286 296 474 496 000}
```

The above values of $\tilde{r}(n)$ can be compared with those computed directly.

In[∘]:=
```
SeqListIniComputation = {};

MAX = 20;

For[n = 0, n ≤ MAX, n++,
  coord = Select[Tuples[Table[i, {i, 0, n}], NN], Total[#] == n &];
  size = Length@coord;
  p = Sum[Multinomial[Sequence @@ (2 coord[[i]])] *
      Product[Binomial[2 n - 2 coord[[i, j]], n - coord[[i, j]]], {j, 1, NN}], {i, 1, size}];
  AppendTo[SeqListIniComputation, p];
 ];

SeqListIniComputation
```

Out[∘]=
```
{1, 32, 6048, 2 451 200, 1 391 236 000, 921 422 380 032, 663 895 856 219 904, 505 041 413 866 868 736,
 399 445 932 990 555 902 880, 325 440 143 503 901 735 429 120, 271 445 584 301 606 582 663 031 808,
 230 773 066 339 125 955 854 130 661 376, 199 326 200 240 673 646 611 787 771 995 904,
 174 478 237 021 099 598 812 491 315 604 889 600, 154 480 035 620 813 053 446 642 174 412 128 768 000,
 138 129 336 609 134 098 952 004 475 839 318 761 472 000,
 124 577 089 053 969 968 356 059 653 140 361 638 344 938 400,
 113 209 463 052 287 193 655 237 025 876 331 530 870 707 737 600,
 103 573 496 015 054 055 969 039 980 718 499 533 706 000 571 520 000,
 95 328 837 240 197 678 160 114 853 748 204 677 385 026 223 109 120 000,
 88 215 610 025 056 975 283 519 690 346 309 846 200 279 286 296 474 496 000}
```

Let us the generate a list of $\tilde{r}(n)$.

In[∘]:=
```
Bound = 5000;

SeqList = UnrollRecurrence[SeqNormalized, Seq[α], SeqListIni, Bound];

seq[n_] := SeqList[[n + 1]];
```

Guess a Minimal ODE for $\tilde{R}(z)$.

Its order is 8, and is identical to that of the ODE in Theorem 4.2 (ODENormalizedinTheta).

In[∘]:=
```
ClearAll[Diff];
ODEGuess = GuessMinDE[Take[SeqList, 300], Diff[z]];
ODEGuessinD = NormalizeCoefficients[
   ToOrePolynomial[ODEGuess /. {Derivative[k_][Diff][z] → Der[z]^k} /. {Diff[z] → 1}]];
```

In[∘]:=
```
ODEGuessinTheta =
  NormalizeCoefficients[ChangeOreAlgebra[z ** ODEGuessinD, OreAlgebra[Euler[z]]]];
```

In[∘]:=
```
ODEGuessinThetaOrder = OrePolynomialDegree[ODEGuessinTheta, Euler[z]]
```

Out[∘]= 8

Compute the asymptotics for $\tilde{r}(n)$.

In[∘]:=
```
AsyList = Asymptotics[SeqNormalized, Seq[α]];
N[AsyList]
```

Out[ ]= $\left\{\frac{16.^{\alpha}}{\alpha^2}, \frac{256.^{\alpha}}{\alpha^2}, \frac{1024.^{\alpha}}{\alpha^3}, \frac{1024.^{\alpha}}{\alpha^2}, \frac{(-871845.)^{\alpha}}{\alpha^9}, \frac{(-2844.77)^{\alpha}}{\alpha^9}, \frac{(-376.522)^{\alpha}}{\alpha^9}, \frac{(-83.424)^{\alpha}}{\alpha^9}, \frac{(-14.7166)^{\alpha}}{\alpha^9}, \frac{0.381565^{\alpha}}{\alpha^9}, \frac{9.72218^{\alpha}}{\alpha^9}, \frac{2293.66^{\alpha}}{\alpha^9}, \frac{(-80.3841 - 13300.8\, i)^{\alpha}}{\alpha^9}, \frac{(-80.3841 + 13300.8\, i)^{\alpha}}{\alpha^9}, \frac{(94.5931 - 184.858\, i)^{\alpha}}{\alpha^9}, \frac{(94.5931 + 184.858\, i)^{\alpha}}{\alpha^9}\right\}$

In[ ]:=
```
Ind = Reverse[Table[Floor[Bound / i], {i, 1, 3}]]
Table[N[seq[Ind[[i]]] / (AsyList[[1]] /. {α → Ind[[i]]})], {i, 1, Length@Ind}]
Table[N[seq[Ind[[i]]] / (AsyList[[2]] /. {α → Ind[[i]]})], {i, 1, Length@Ind}]
Table[N[seq[Ind[[i]]] / (AsyList[[3]] /. {α → Ind[[i]]})], {i, 1, Length@Ind}]
Table[N[seq[Ind[[i]]] / (AsyList[[4]] /. {α → Ind[[i]]})], {i, 1, Length@Ind}]
Table[N[seq[Ind[[i]]] / (AsyList[[11]] /. {α → Ind[[i]]})], {i, 1, Length@Ind}]
Table[N[seq[Ind[[i]]] / (AsyList[[12]] /. {α → Ind[[i]]})], {i, 1, Length@Ind}]
```

Out[ ]= {1666, 2500, 5000}

Out[ ]= $\{2.806687457612096 \times 10^{3007}, 6.343600724639624 \times 10^{4513}, 1.787780641892824 \times 10^{9029}\}$

Out[ ]= $\{2.422718463768892 \times 10^{1001}, 3.179649140402995 \times 10^{1503}, 4.491598734476526 \times 10^{3008}\}$

Out[ ]= {37.5001, 56.2783, 112.568}

Out[ ]= {0.0225091, 0.0225113, 0.0225136}

General: $\frac{1}{9.72218^{1666}}$ is too small to represent as a normalized machine number; precision may be lost.

General: $\frac{1}{9.72218^{2500}}$ is too small to represent as a normalized machine number; precision may be lost.

General: $\frac{1}{9.72218^{5000}}$ is too small to represent as a normalized machine number; precision may be lost.

General: Further output of General::munfl will be suppressed during this calculation.

Out[ ]= {0., 0., 0.}

General: $\frac{1}{2293.66^{1666}}$ is too small to represent as a normalized machine number; precision may be lost.

General: $\frac{1}{2293.66^{2500}}$ is too small to represent as a normalized machine number; precision may be lost.

General: $\frac{1}{2293.66^{5000}}$ is too small to represent as a normalized machine number; precision may be lost.

General: Further output of General::munfl will be suppressed during this calculation.

Out[∘]= {0., 0., 0.}

Approximate the Polya number.

In[∘]:= $\texttt{AtOne} = \texttt{N}\left[\texttt{Sum}\left[\texttt{seq[n]} * \left(\frac{1}{2^{\texttt{MM}}\,\texttt{Binomial[NN, MM]}}\right)^{2\,n}, \{n, 0, \texttt{Bound}\}\right], 11\right]$

$\texttt{N}\left[1 - \frac{1}{\texttt{AtOne}}, 10\right]$

Out[∘]= 1.0452834156

Out[∘]= 0.04332166274

Display the ODE for $\tilde{R}(z)$ in Theorem 4.2

In[∘]:= `ODENormalizedinTheta /. {w → z}`

Out[∘]=

$$
\begin{aligned}
&\big(-756 - 658\,107\,072\, z + 749\,920\,296\,960\, z^2 - 111\,850\,497\,389\,887\,488\, z^3 + \\
&\quad 166\,498\,086\,762\,886\,201\,344\, z^4 + 781\,156\,297\,810\,381\,520\,240\,640\, z^5 - \\
&\quad 1\,637\,147\,560\,168\,901\,326\,135\,099\,392\, z^6 + 732\,231\,540\,023\,730\,620\,969\,986\,818\,048\, z^7 + \\
&\quad 149\,697\,886\,463\,404\,317\,932\,617\,973\,366\,784\, z^8 - 96\,719\,208\,505\,536\,419\,841\,142\,621\,892\,247\,552\, z^9 + \\
&\quad 17\,986\,272\,310\,903\,846\,816\,671\,667\,502\,362\,656\,768\, z^{10} - \\
&\quad 1\,365\,121\,772\,758\,889\,406\,361\,975\,817\,513\,893\,625\,856\, z^{11} - \\
&\quad 266\,982\,380\,934\,205\,139\,034\,767\,194\,888\,213\,610\,102\,784\, z^{12} + \\
&\quad 3\,451\,537\,920\,763\,342\,815\,087\,344\,036\,232\,525\,206\,519\,808\, z^{13} + \\
&\quad 57\,677\,198\,298\,608\,369\,079\,887\,772\,175\,160\,628\,008\,189\,952\, z^{14} - \\
&\quad 655\,787\,571\,926\,373\,008\,384\,765\,243\,492\,546\,204\,588\,310\,528\, z^{15} + \\
&\quad 241\,642\,117\,251\,606\,275\,763\,798\,810\,128\,911\,651\,647\,258\,624\, z^{16}\big)\, \theta_z^8 + \\
&\big(1512 + 1\,985\,050\,368\, z + 6\,350\,035\,230\,720\, z^2 + 557\,203\,438\,524\,432\,384\, z^3 + \\
&\quad 587\,179\,135\,837\,113\,679\,872\, z^4 - 6\,373\,155\,470\,045\,165\,823\,983\,616\, z^5 + \\
&\quad 1\,292\,611\,484\,404\,060\,407\,000\,465\,408\, z^6 + 5\,768\,579\,650\,367\,308\,639\,843\,349\,692\,416\, z^7 - \\
&\quad 1\,382\,427\,013\,757\,614\,047\,365\,601\,869\,955\,072\, z^8 - 358\,992\,894\,157\,107\,285\,438\,601\,960\,070\,578\,176\, z^9 + \\
&\quad 53\,119\,839\,081\,791\,910\,699\,130\,795\,605\,268\,365\,312\, z^{10} - \\
&\quad 20\,033\,688\,987\,472\,672\,410\,340\,587\,286\,921\,796\,911\,104\, z^{11} - \\
&\quad 1\,070\,184\,586\,795\,418\,191\,307\,064\,402\,944\,853\,050\,130\,432\, z^{12} + \\
&\quad 47\,857\,531\,084\,168\,298\,968\,105\,213\,313\,896\,536\,229\,281\,792\, z^{13} + \\
&\quad 270\,884\,195\,420\,762\,288\,774\,538\,561\,958\,062\,360\,455\,282\,688\, z^{14} - \\
&\quad 5\,786\,930\,575\,834\,638\,459\,669\,201\,820\,980\,856\,382\,306\,648\,064\, z^{15} + \\
&\quad 1\,933\,136\,938\,012\,850\,206\,110\,390\,481\,031\,293\,213\,178\,068\,992\, z^{16}\big)\, \theta_z^7 + \\
&\big(-1113 - 1\,950\,259\,584\, z - 6\,216\,682\,061\,824\, z^2 - 988\,841\,093\,180\,162\,048\, z^3 - \\
&\quad 1\,233\,270\,686\,793\,691\,299\,840\, z^4 + 9\,715\,936\,946\,399\,911\,434\,256\,384\, z^5 + \\
&\quad 7\,752\,074\,094\,169\,934\,619\,780\,055\,040\, z^6 + 3\,292\,303\,250\,603\,115\,641\,274\,504\,314\,880\, z^7 - \\
&\quad 6\,432\,398\,522\,881\,878\,319\,620\,478\,709\,268\,480\, z^8 - 1\,442\,519\,265\,201\,053\,822\,094\,460\,496\,124\,051\,456\, z^9 + \\
&\quad 81\,315\,393\,847\,369\,615\,391\,288\,576\,991\,201\,067\,008\, z^{10} - \\
&\quad 46\,854\,607\,085\,100\,541\,227\,643\,541\,228\,521\,577\,775\,104\, z^{11} - \\
&\quad 484\,722\,323\,648\,843\,742\,960\,229\,713\,378\,845\,655\,040\,000\, z^{12} + \\
&\quad 230\,657\,489\,060\,094\,958\,856\,963\,994\,975\,994\,903\,435\,149\,312\, z^{13} + \\
&\quad 720\,571\,084\,999\,990\,630\,257\,768\,886\,154\,063\,035\,548\,827\,648\, z^{14} - \\
&\quad 22\,027\,520\,447\,398\,165\,760\,511\,055\,419\,885\,169\,351\,394\,852\,864\, z^{15} + \\
&\quad 6\,663\,616\,997\,264\,781\,396\,236\,424\,132\,096\,584\,504\,800\,444\,416\, z^{16}\big)\, \theta_z^6 + \\
&\big(357 + 789\,151\,104\, z + 10\,558\,964\,416\,512\, z^2 + 679\,933\,490\,467\,176\,448\, z^3 + 287\,227\,915\,264\,289\,931\,264\, z^4 - \\
&\quad 14\,427\,253\,172\,957\,536\,013\,254\,656\, z^5 - 3\,092\,284\,696\,452\,480\,308\,007\,141\,376\, z^6 - \\
&\quad 905\,311\,608\,923\,360\,926\,047\,701\,827\,584\, z^7 - 14\,621\,774\,397\,415\,013\,636\,807\,083\,954\,274\,304\, z^8 -
\end{aligned}
$$

$$1520148020883568461909863756948570112\,z^{9}+$$
$$386791883303174384286527316852952006656\,z^{10}-$$
$$55223977247937670737556473181100776095744\,z^{11}+$$
$$3961198864716838655960160400530693479727104\,z^{12}+$$
$$595909152288030158390074172187987674363068416\,z^{13}+$$
$$1344613983895642776006711946247382448056827904\,z^{14}-$$
$$46737381677309460398827589903959671231607734272\,z^{15}+$$
$$12917784851408785491873078058141402044309700608\,z^{16}\Big)\,\theta_z^{5}+$$
$$\Big(-42-103505472\,z-4717152813056\,z^{2}-229951271138492416\,z^{3}-$$
$$33452654058350313472\,z^{4}+3519629264891117955973120\,z^{5}+$$
$$8712355168877862347467653120\,z^{6}-3183462774294546535677280911360\,z^{7}-$$
$$12662024101532041005416571287371776\,z^{8}-91457574708638075983201720533516288\,z^{9}+$$
$$586373716393067719463798499745499971584\,z^{10}-$$
$$45402712266053628419392613379787913691136\,z^{11}+$$
$$9548218855973838530825534106648111229173760\,z^{12}+$$
$$910171319762953713098938394074694947157573632\,z^{13}+$$
$$1695248459973650411462298355247964543229362176\,z^{14}-$$
$$60316818440945087853828024483516860568289935360\,z^{15}+$$
$$15391679930285039325517386362534096505705332736\,z^{16}\Big)\,\theta_z^{4}+$$
$$\Big(212352\,z+41049243648\,z^{2}-8757517736738816\,z^{3}+20173834021513461760\,z^{4}-$$
$$285697925187496921006080\,z^{5}+5192831041959280753355259904\,z^{6}-$$
$$1854167396972337514541117079552\,z^{7}-7611731718211226366541506826731520\,z^{8}+$$
$$1132092616093392427901870092654215168\,z^{9}+$$
$$573360955845607449871633552338403196928\,z^{10}-$$
$$31196974018934147496719981299967906021376\,z^{11}+$$
$$10343484480536631324792615211856556940328960\,z^{12}+$$
$$850871471160179197799997784539641065269886976\,z^{13}+$$
$$1366110491586634438250598685972842495483052032\,z^{14}-$$
$$48462627453914171613580503834409896539840315392\,z^{15}+$$
$$11533376887988124536976314041777845706747281408\,z^{16}\Big)\,\theta_z^{3}+$$
$$\Big(102144\,z-45890052096\,z^{2}-4372668181905408\,z^{3}+25373328015678767104\,z^{4}+$$
$$35860603273980739059712\,z^{5}+2609042215039715330989490176\,z^{6}-$$
$$1135805724897588664940548325376\,z^{7}-2255387710140891706830918298632192\,z^{8}+$$
$$1208690199949684174443411490448867328\,z^{9}+$$
$$334887474030943944488261929148495167488\,z^{10}-$$
$$17877519858996120053115971187944045150208\,z^{11}+$$
$$6128654166961763785820170570933495910105088\,z^{12}+$$
$$476288752718822257140265748951023617535115264\,z^{13}+$$
$$668324955523996949091967282097867454465179648\,z^{14}-$$
$$23673396984425987991182604909788067572173766656\,z^{15}+$$
$$5303121535030477312378786039652035049432285184\,z^{16}\Big)\,\theta_z^{2}+$$
$$\Big(18816\,z-15679168512\,z^{2}-1105852812492800\,z^{3}+9253977260438847488\,z^{4}+$$
$$36631485914913630584832\,z^{5}+726314268655758437624315904\,z^{6}-$$
$$358748918263218897800795258880\,z^{7}-286918040829362957086349485670400\,z^{8}+$$
$$503093135988065408878446537502359552\,z^{9}+$$
$$108054624128516395031347156140800606208\,z^{10}-$$
$$6462199176714597967385137880595550961664\,z^{11}+$$
$$1918694308581208774434293647882766231011328\,z^{12}+$$
$$145968214956821518855061140764657290013835264\,z^{13}+$$
$$180192916196793417299520478544551103988498432\,z^{14}-$$
$$6427990896954765589117223123312617887797084160\,z^{15}+$$

$1\,366\,788\,225\,704\,397\,997\,288\,987\,019\,791\,656\,529\,629\,806\,592\,z^{16}\big)\,\theta_z +$

$\big(1344\,z - 1\,639\,464\,960\,z^2 - 114\,267\,116\,273\,664\,z^3 + 1\,211\,269\,289\,902\,866\,432\,z^4 +$

$5\,825\,052\,469\,481\,755\,901\,952\,z^5 + 84\,152\,329\,059\,287\,491\,751\,706\,624\,z^6 -$

$48\,938\,139\,253\,071\,191\,076\,992\,188\,416\,z^7 - 3\,045\,898\,181\,345\,513\,899\,617\,530\,413\,056\,z^8 +$

$78\,022\,182\,208\,697\,643\,235\,066\,215\,175\,028\,736\,z^9 +$

$14\,678\,268\,634\,598\,917\,861\,557\,009\,824\,329\,236\,480\,z^{10} -$

$991\,390\,991\,530\,383\,611\,754\,057\,315\,362\,342\,436\,864\,z^{11} +$

$247\,958\,505\,832\,498\,167\,951\,336\,010\,415\,935\,397\,560\,320\,z^{12} +$

$18\,747\,996\,529\,475\,474\,000\,600\,656\,049\,610\,020\,358\,193\,152\,z^{13} +$

$20\,499\,222\,726\,707\,352\,515\,629\,191\,626\,716\,497\,397\,678\,080\,z^{14} -$

$742\,685\,376\,897\,284\,273\,453\,811\,376\,847\,779\,469\,564\,313\,600\,z^{15} +$

$151\,026\,323\,282\,253\,922\,352\,374\,256\,330\,569\,782\,279\,536\,640\,z^{16}\big)$