Multi-headed Lattice Green Function (N = 5, M = 2)
Find Minimal REC

```
In[∘]:= NN = 5;
MM = 2;
```

Recall some basic definitions in the paper:

$$P_{M,N}(z) := \frac{1}{(2\pi)^N} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{1}{1 - \frac{z}{\binom{N}{M}} \sigma_M(\cos\theta_1, \ldots, \cos\theta_N)} d\theta_1 \ldots d\theta_N$$

$$R_{M,N}(z) := P_{M,N}\left(2^M \binom{N}{M} z\right) \text{ and } R_{M,N}(z) = \sum_{n\geq 0} r_{M,N}(n)\, z^n$$

Also, for M odd or $M = N$, we always have $r(2n+1) = 0$. Hence, define
$\tilde{r}_{M,N}(n) := r_{M,N}(2n)$ and $\tilde{R}_{M,N}(z) := \sum_{n\geq 0} \tilde{r}_{M,N}(n)\, z^n = \sum_{n\geq 0} r_{M,N}(2n)\, z^n$

Our goal is to find:

Case 1. *M* even and *M* ≠ *N*:
- minimal recurrences (REC) for $r(n)$.

Case 2. *M* odd or *M* = *N*:
- minimal recurrences (REC) for $\tilde{r}(n)$.

Command: UnrollRecurrence

Generate a sequence from recurrence & initial values (Koutschan's implementation).

```
In[∘]:= (* given a recurrence rec in f[n], compute the values {f[0],f[1],...,f[bound]}
 where inits are the initial values
 {f[0],...,f[d-1]} with d being the order of the recurrence *)
Clear[UnrollRecurrence];
UnrollRecurrence[rec1_, f_[n_], inits_, bound_] :=
  Module[{i, x, vals = inits, rec = rec1},
   If[Head[rec] =!= Equal, rec = (rec == 0)];
   rec = rec /. n → n - Max[Cases[rec, f[n + a_.] :→ a, Infinity]];
   Do[
    AppendTo[vals,
      Solve[rec /. n → i /. f[i] → x /. f[a_] :→ vals[[a + 1]], x][[1, 1, 2]]];
    , {i, Length[inits], bound}];
   Return[vals];
  ];
```

Load RISC packages.

```
In[•]:= << RISC`HolonomicFunctions`
<< RISC`Asymptotics`
<< RISC`Guess`
```



```
--> Type  ?HolonomicFunctions  for help.
```


We start by importing known ODE for $R(z)$.

Note that the ODE in Koutschan (2013, pp. 11-12, Thm 3) is for $P(z) = R\left(z \middle/ \binom{N}{M} 2^M\right)$.

```
In[∘]:= ODEDiv2 = ToOrePolynomial[
    30 * (27 000 000 000 + 84 037 500 000 * z - 346 865 625 000 * z^2 - 55 567 000 000 * z^3 +
          187 923 165 625 * z^4 + 36 477 006 875 * z^5 + 21 336 230 625 * z^6 + 19 123 388 575 * z^7 +
          6 925 739 310 * z^8 + 1 443 544 710 * z^9 + 163 913 184 * z^10 + 7 525 440 * z^11) * P[z] +
      10 * (-189 000 000 000 + 4 816 462 500 000 * z - 7 268 326 875 000 * z^2 -
          21 210 430 812 500 * z^3 + 2 664 478 321 875 * z^4 +
          3 711 617 481 250 * z^5 - 135 661 728 250 * z^6 + 689 643 286 650 * z^7 +
          607 021 304 825 * z^8 + 209 673 119 160 * z^9 + 40 678 130 502 * z^10 +
          4 143 853 440 * z^11 + 167 064 768 * z^12) * Derivative[1][P][z] +
      5 * (-3 240 000 000 000 + 5 055 750 000 000 * z + 44 457 862 500 000 * z^2 -
          133 825 053 750 000 * z^3 - 110 925 736 437 500 * z^4 + 13 367 806 743 750 * z^5 -
          6 199 228 765 625 * z^6 - 8 282 515 456 375 * z^7 + 1 646 226 060 075 * z^8 +
          2 287 368 823 475 * z^9 + 810 956 145 330 * z^10 + 149 186 684 934 * z^11 +
          13 819 981 248 * z^12 + 496 679 040 * z^13) * Derivative[2][P][z] +
      5 * z * (-13 162 500 000 000 + 45 343 125 000 000 * z + 40 530 375 000 000 * z^2 -
          190 176 960 000 000 * z^3 - 77 498 059 625 000 * z^4 - 3 649 915 059 375 * z^5 -
          26 918 293 320 000 * z^6 - 13 545 524 756 500 * z^7 - 465 440 555 100 * z^8 +
          1 350 059 072 325 * z^9 + 524 857 986 060 * z^10 + 92 744 995 638 * z^11 +
          7 892 060 544 * z^12 + 255 864 960 * z^13) * Derivative[3][P][z] +
      10 * z^2 * (-5 568 750 000 000 + 23 905 125 000 000 * z + 3 393 646 875 000 * z^2 -
          39 702 348 750 000 * z^3 - 7 716 298 734 375 * z^4 - 3 779 011 321 875 * z^5 -
          7 801 785 421 250 * z^6 - 3 351 125 770 500 * z^7 - 382 134 335 775 * z^8 +
          148 313 757 125 * z^9 + 68 439 921 540 * z^10 + 11 725 276 842 * z^11 +
          923 795 772 * z^12 + 27 279 720 * z^13) * Derivative[4][P][z] +
      8 * z^3 * (5 + z) * (-354 375 000 000 + 1 774 828 125 000 * z - 503 550 000 000 * z^2 -
          1 289 447 109 375 * z^3 + 254 876 515 625 * z^4 - 266 627 903 125 * z^5 -
          304 623 830 625 * z^6 - 87 265 479 875 * z^7 - 4 878 146 975 * z^8 + 3 939 663 705 * z^9 +
          1 048 560 285 * z^10 + 97 471 734 * z^11 + 3 057 210 * z^12) * Derivative[5][P][z] +
      16 * (-5 + z) * (-1 + z) * z^4 * (5 + z)^2 * (10 + z) * (15 + z) * (5 + 3 * z) *
        (-675 000 + 3 465 000 * z - 1 053 375 * z^2 + 933 650 * z^3 +
          449 735 * z^4 + 144 776 * z^5 + 15 678 * z^6) * Derivative[6][P][z] /.
    {Derivative[k_][P][z] → Der[z]^k} /. {P[z] → 1}];
```

Process the data.

Write the ODE in terms of the operators D and θ.

```
In[∘]:= ODENormalizedinD = NormalizeCoefficients[DFiniteSubstitute[{ODEDiv2},
      {z → w * 2^MM * Binomial[NN, MM]}, Algebra → OreAlgebra[Der[w]]][[1]]];
```

```
In[∘]:= ODENormalizedinTheta =
    NormalizeCoefficients[ChangeOreAlgebra[w ** ODENormalizedinD, OreAlgebra[Euler[w]]]];
```

Then transform the above to a REC for $r(n)$ and write it explicitly.

```
In[∘]:= RECNormalizedinS =
    NormalizeCoefficients[DFiniteDE2RE[{ODENormalizedinD}, {w}, {α}][[1]]];
```

```
In[∘]:= RECNormalizedinSOrder = OrePolynomialDegree[RECNormalizedinS, S[α]]

Out[∘]= 13
```

```
In[∘]:= ClearAll[Seq];
SeqNormalized = ApplyOreOperator[RECNormalizedinS, Seq[α]];
```

Compute the initial values of $r(n)$.

In[∘]:=
```
MAX = RECNormalizedinSOrder;
ClearAll[a];

SeriesIni = ApplyOreOperator[ODENormalizedinTheta, Sum[a[n] w^n, {n, 0, MAX}]];

SeriesIniSol = Solve[Join[Table[Coefficient[SeriesIni, w, i] == 0, {i, 1, MAX}],
   {a[0] == 1}, {a[1] == 0}], Table[a[i], {i, 0, MAX}]]

SeqListIni = Table[SeriesIniSol[[1, k, 2]], {k, 1, Length@SeriesIniSol[[1]]}]

seq[n_] := SeqListIni[[n + 1]];
```

Out[∘]=
```
{{a[0] → 1, a[1] → 0, a[2] → 40, a[3] → 480, a[4] → 11880, a[5] → 281280, a[6] → 7506400,
  a[7] → 210268800, a[8] → 6166993000, a[9] → 187069411200, a[10] → 5833030976640,
  a[11] → 186014056166400, a[12] → 6044435339896800, a[13] → 199561060892793600}}
```

Out[∘]=
```
{1, 0, 40, 480, 11880, 281280, 7506400, 210268800, 6166993000, 187069411200,
 5833030976640, 186014056166400, 6044435339896800, 199561060892793600}
```

Generate a list of $r(n)$.

In[∘]:=
```
Bound = 200;

SeqList = UnrollRecurrence[SeqNormalized, Seq[α], SeqListIni, Bound];

seq[n_] := SeqList[[n + 1]];
```

Guess a Minimal REC for $r(n)$.

SeqfromRECGuess gives the REC in Theorem 6.2! (To be displayed at the end of this notebook)

REC: Order 7

ODE: Order 19, Degree 7

In[∘]:=
```
ClearAll[Seq];
RECGuess = GuessMinRE[Take[SeqList, 200], Seq[α]];
RECGuessinS = NormalizeCoefficients[ToOrePolynomial[RECGuess /. {Seq[k_] → S[α]^(k-α)}]];
```

In[∘]:=
```
RECGuessinSOrder = OrePolynomialDegree[RECGuessinS, S[α]]
```

Out[∘]= 7

In[∘]:=
```
ODEfromRECGuessinD =
  NormalizeCoefficients[DFiniteRE2DE[{RECGuessinS}, {α}, {z}][[1]]];
```

In[∘]:=
```
ODEfromRECGuessinTheta = NormalizeCoefficients[
   ChangeOreAlgebra[z ** ODEfromRECGuessinD, OreAlgebra[Euler[z]]]];
```

In[∘]:=
```
ODEfromRECGuessinThetaOrder = OrePolynomialDegree[ODEfromRECGuessinTheta, Euler[z]]
```

Out[∘]= 19

In[∘]:=
```
ODEfromRECGuessinThetaDegree =
 Max[Exponent[OrePolynomialListCoefficients[ODEfromRECGuessinTheta], z]]
```

Out[∘]= 7

We may also write this REC explicitly.

In[∘]:=
```
ClearAll[Seq];
SeqfromRECGuess = ApplyOreOperator[RECGuessinS, Seq[α]];
```

In[∘]:=
```
SeqfromRECGuessList =
  UnrollRecurrence[SeqfromRECGuess, Seq[α], Take[SeqList, RECGuessinSOrder], 200];
```

Prove the minimal REC for $r(n)$.

In[∘]:=
```
RECCompare = DFinitePlus[{RECNormalizedinS}, {RECGuessinS}][[1]];
```

Compute the *largest* positive integral root of the leading coefficient in the recurrence RECCompare.

In[∘]:=
```
LeadCoeff = RECCompare[[1, 1, 1]];
LeadCoeffRoot = Solve[LeadCoeff == 0, α][[All, 1, 2]]
```

Out[∘]= {-13, -13, -13, -13, -13, -12}

There are no positive integral roots in our case.

In[∘]:=
```
Select[Select[LeadCoeffRoot, IntegerQ], # > 0 &]
```

Out[∘]= { }

In[∘]:=
```
RECCompareOrder = OrePolynomialDegree[RECCompare, S[α]]
```

Out[∘]= 13

In[∘]:=
```
CheckNum = RECCompareOrder + 20;
Take[SeqList, CheckNum] - Take[SeqfromRECGuessList, CheckNum]
```

Out[∘]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

Display the REC in Theorem 6.2

In[∘]:=
```
Collect[Expand[SeqfromRECGuess], Seq[_]]
```

Out[ ]= $(42\,140\,738\,676\,326\,400\,000 + 157\,842\,901\,249\,818\,624\,000\,\alpha +$
$257\,331\,505\,709\,737\,574\,400\,\alpha^{2} + 243\,764\,108\,399\,982\,673\,920\,\alpha^{3} +$
$150\,397\,023\,447\,243\,816\,960\,\alpha^{4} + 63\,968\,341\,924\,254\,842\,880\,\alpha^{5} + 19\,301\,263\,998\,729\,584\,640\,\alpha^{6} +$
$4\,174\,508\,253\,346\,529\,280\,\alpha^{7} + 643\,779\,101\,841\,162\,240\,\alpha^{8} + 69\,168\,932\,868\,587\,520\,\alpha^{9} +$
$4\,922\,454\,740\,828\,160\,\alpha^{10} + 208\,614\,614\,630\,400\,\alpha^{11} + 3\,986\,266\,521\,600\,\alpha^{12})$ Seq[α] +
$(118\,427\,858\,324\,029\,440\,000 + 355\,246\,559\,316\,108\,902\,400\,\alpha + 481\,552\,669\,599\,250\,186\,240\,\alpha^{2} +$
$390\,301\,079\,007\,991\,857\,152\,\alpha^{3} + 210\,764\,527\,991\,633\,575\,936\,\alpha^{4} +$
$79\,918\,506\,618\,774\,847\,488\,\alpha^{5} + 21\,826\,970\,852\,964\,532\,224\,\alpha^{6} +$
$4\,327\,696\,049\,218\,387\,968\,\alpha^{7} + 618\,429\,092\,691\,574\,784\,\alpha^{8} + 62\,134\,020\,238\,999\,552\,\alpha^{9} +$
$4\,167\,373\,533\,741\,056\,\alpha^{10} + 167\,578\,215\,383\,040\,\alpha^{11} + 3\,056\,137\,666\,560\,\alpha^{12})$ Seq[$1 + \alpha$] +
$(62\,676\,619\,662\,919\,680\,000 + 168\,213\,967\,990\,385\,049\,600\,\alpha + 205\,820\,392\,167\,964\,974\,080\,\alpha^{2} +$
$151\,791\,584\,110\,964\,534\,272\,\alpha^{3} + 75\,137\,340\,688\,642\,841\,600\,\alpha^{4} +$
$26\,295\,232\,911\,598\,126\,080\,\alpha^{5} + 6\,670\,149\,766\,003\,083\,264\,\alpha^{6} +$
$1\,235\,525\,904\,487\,723\,008\,\alpha^{7} + 165\,841\,646\,014\,996\,480\,\alpha^{8} + 15\,729\,900\,132\,270\,080\,\alpha^{9} +$
$1\,000\,638\,108\,860\,416\,\alpha^{10} + 38\,329\,059\,901\,440\,\alpha^{11} + 668\,530\,114\,560\,\alpha^{12})$ Seq[$2 + \alpha$] +
$(1\,794\,185\,247\,360\,768\,000 + 4\,260\,839\,636\,091\,043\,840\,\alpha + 4\,649\,746\,903\,477\,813\,888\,\alpha^{2} +$
$3\,082\,953\,754\,682\,083\,328\,\alpha^{3} + 1\,382\,952\,049\,413\,254\,272\,\alpha^{4} + 442\,032\,317\,052\,873\,728\,\alpha^{5} +$
$103\,190\,706\,316\,889\,344\,\alpha^{6} + 17\,720\,524\,544\,509\,952\,\alpha^{7} + 2\,220\,812\,336\,954\,368\,\alpha^{8} +$
$198\,014\,286\,036\,992\,\alpha^{9} + 11\,919\,389\,769\,728\,\alpha^{10} + 434\,786\,795\,520\,\alpha^{11} + 7\,266\,631\,680\,\alpha^{12})$
Seq[$3 + \alpha$] + $(-3\,522\,851\,180\,688\,416\,000 - 8\,446\,568\,365\,407\,735\,680\,\alpha -$
$9\,248\,095\,565\,260\,356\,576\,\alpha^{2} - 6\,114\,775\,140\,268\,882\,576\,\alpha^{3} -$
$2\,719\,484\,985\,845\,017\,792\,\alpha^{4} - 857\,108\,315\,069\,629\,104\,\alpha^{5} - 196\,310\,820\,429\,867\,616\,\alpha^{6} -$
$32\,924\,151\,546\,376\,000\,\alpha^{7} - 4\,013\,146\,001\,886\,336\,\alpha^{8} - 346\,719\,870\,364\,160\,\alpha^{9} -$
$20\,154\,401\,039\,360\,\alpha^{10} - 707\,739\,648\,000\,\alpha^{11} - 11\,354\,112\,000\,\alpha^{12})$ Seq[$4 + \alpha$] +
$(-458\,904\,717\,778\,020\,000 - 1\,056\,134\,626\,035\,848\,800\,\alpha - 1\,109\,896\,707\,061\,337\,856\,\alpha^{2} -$
$704\,344\,314\,090\,018\,780\,\alpha^{3} - 300\,647\,030\,233\,781\,612\,\alpha^{4} - 90\,944\,593\,157\,694\,708\,\alpha^{5} -$
$19\,993\,089\,019\,041\,540\,\alpha^{6} - 3\,218\,776\,240\,146\,608\,\alpha^{7} - 376\,681\,142\,235\,984\,\alpha^{8} -$
$31\,252\,297\,558\,272\,\alpha^{9} - 1\,745\,103\,671\,296\,\alpha^{10} - 58\,889\,994\,240\,\alpha^{11} - 908\,328\,960\,\alpha^{12})$ Seq[$5 + \alpha$] +
$(-1\,106\,658\,753\,555\,600 - 2\,330\,306\,062\,592\,328\,\alpha - 2\,249\,741\,897\,564\,436\,\alpha^{2} -$
$1\,317\,143\,965\,540\,014\,\alpha^{3} - 520\,970\,340\,108\,810\,\alpha^{4} - 146\,691\,130\,015\,168\,\alpha^{5} -$
$30\,156\,685\,922\,334\,\alpha^{6} - 4\,561\,556\,620\,082\,\alpha^{7} - 503\,951\,197\,636\,\alpha^{8} -$
$39\,663\,617\,640\,\alpha^{9} - 2\,111\,344\,496\,\alpha^{10} - 68\,259\,840\,\alpha^{11} - 1\,013\,760\,\alpha^{12})$ Seq[$6 + \alpha$] +
$(836\,209\,651\,013\,100 + 1\,823\,470\,291\,632\,528\,\alpha + 1\,811\,702\,917\,816\,029\,\alpha^{2} +$
$1\,084\,613\,257\,235\,718\,\alpha^{3} + 435\,833\,439\,807\,171\,\alpha^{4} + 123\,860\,858\,052\,324\,\alpha^{5} +$
$25\,531\,982\,914\,119\,\alpha^{6} + 3\,847\,089\,898\,422\,\alpha^{7} + 420\,608\,699\,769\,\alpha^{8} +$
$32\,547\,074\,928\,\alpha^{9} + 1\,692\,297\,492\,\alpha^{10} + 53\,095\,680\,\alpha^{11} + 760\,320\,\alpha^{12})$ Seq[$7 + \alpha$]