KKS2025, Conjecture 23, eq. (10.8)

In[∘]:=
```
<< RISC`HolonomicFunctions`;
<< RISC`Guess`;
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
--> Type  ?HolonomicFunctions  for help.
```

Package GeneratingFunctions version 0.9 written by Christian Mallinger
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

Guess Package version 0.52
written by Manuel Kauers
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

In[∘]:=
```
SetDirectory[NotebookDirectory[]];
```

The following initializing codes are taken from Christoph Koutschan, Christian Krattenthaler and Michael Schlosser's implementation for their 2025 JSC paper on determinant evaluations.
http://www.koutschan.de/data/det3/

Reference

C. Koutschan, C. Krattenthaler, and M. J. Schlosser, Determinant evaluations inspired by Di Francesco's determinant for twenty-vertex configurations, *J. Symbolic Comput.* **127** (2025), Paper No. 102352, 34 pp.
https://doi.org/10.1016/j.jsc.2024.102352

In[∘]:=
```
(* Display all relevant information about an annihilator ideal. *)
AnnInfo[ann_] := With[{vars = First /@ OreAlgebra[ann][[1]]}, Print[
    "ByteCount: ", ByteCount[ann],
    "\nSupport: ", Support[ann],
    "\ndegree " <> ToString[vars] <> ": ", Exponent[#, vars] & /@ ann,
    "\nStandard Monomials: ", UnderTheStaircase[ann],
    "\nHolonomic Rank: ", Length[UnderTheStaircase[ann]]
   ]];
```

In[•]:=
```
(* A straight-
 forward implementation of reduction modulo a left ideal in the shift algebra. *)
(* Reason: the built-in procedure "OreReduce"
   in the HolonomicFunctions package sometimes
   causes Mathematica to crash. *)
SortLex[m1_, m2_] := With[{f1 = First[m1], f2 = First[m2]},
   If[f1 =!= f2 || Length[m1] === 1, f1 > f2, SortLex[Rest[m1], Rest[m2]]]];
SortDLex[m1_, m2_] := With[{w1 = Plus @@ m1, w2 = Plus @@ m2},
   If[w1 === w2, SortLex[m1, m2], w1 > w2]];
Add[p1_List, p2_List] :=
  Module[{p = {}, c, i1 = 1, i2 = 1, l1 = Length[p1], l2 = Length[p2], e1, e2},
   While[i1 ≤ l1 && i2 ≤ l2,
    {e1, e2} = {p1[[i1, 2]], p2[[i2, 2]]};
    Which[
     e1 === e2, If[(c = p1[[i1, 1]] + p2[[i2, 1]]) =!= 0, AppendTo[p, {c, e1}]];
     i1++; i2++;
     ,
     SortDLex[e1, e2], AppendTo[p, p1[[i1]]]; i1++;
     ,
     SortDLex[e2, e1], AppendTo[p, p2[[i2]]]; i2++;
    ];
   ];
   If[i1 ≤ l1, p = Join[p, Take[p1, {i1, l1}]]];
   If[i2 ≤ l2, p = Join[p, Take[p2, {i2, l2}]]];
   Return[p];
  ];
ScalarMult[s_, p_List] := {Expand[Together[s * #1]], #2} & @@@ p;
OreReduce1[p_List, g_List] := OreReduce1[#, g] & /@ p;
OreReduce1[p1_OrePolynomial, g1 : {(_OrePolynomial) ..}] :=
  Module[{p = p1, g = g1, v, e, f, f1, r = {}, k, gk, gcd},
   v = First /@ OreAlgebra[p][[1]];
   {p, g} = {First[p], First /@ g};
   f = PolynomialLCM @@ (Denominator[First[#]] & /@ p);
   p = ScalarMult[f, p];
   While[p =!= {},
    k = 1;
    While[Min[e = (p[[1, 2]] - g[[k, 1, 2]])] < 0, k++];
    If[k > Length[g],
     AppendTo[r, p[[1]]];
     p = Rest[p];
     ,
     gk = {Expand[#1 /. Thread[v → (v + e)]], #2 + e} & @@@ g[[k]];
     gcd = PolynomialGCD[p[[1, 1]], gk[[1, 1]]];
     f *= (f1 = Together[gk[[1, 1]] / gcd]);
     gk = ScalarMult[Together[-p[[1, 1]] / gcd], Rest[gk]];
     p = Add[ScalarMult[f1, Rest[p]], gk];
    ];
   ];
   Return[OrePolynomial[{Together[#1 / f], #2} & @@@ r, p1[[2]], p1[[3]]]];
  ];
```

In[ ]:=
```
ClearAll[prod];

prodsimp = {prod[a_, {i_, b_}] → prod[a, {i, 1, b}],
   prod[a_, {i_, b0_, b1_}] / prod[a_, {i_, b0_, b2_}] /; IntegerQ[Expand[b1 - b2]] :>
    If[Expand[b1 - b2] ≥ 0, Product[a, {i, b2 + 1, b1}], 1 / Product[a, {i, b1 + 1, b2}]],
   prod[a1_, b_]^e1_. * prod[a2_, b_]^e2_. :> prod[FunctionExpand[a1^e1 * a2^e2], b]};
```

Initialization

Set up the determinant (of matrix $a_{i,j}$) in question.

In[ ]:=
```
ClearAll[mata, mata1, mata2, matc, datac, prodform];
```

In[ ]:=
```
ClearAll[a, b, c, d, e, f, i, j, n];

Print["We are going to evaluate the determinant:\n",
  TraditionalForm[HoldForm @@ {Subscript[det, 0 ≤ i, j < n][
      e^(i + b) Binomial[f * j + i + c, f * j + a] + Binomial[f * j - i + d, f * j + a]]}], "\n"];

{a, b, c, d} = {3, 0, 3, 3};
{e, f} = {2, 4};

mata1[i_, j_] := e^(i + b) Binomial[f * j + i + c, f * j + a];
mata2[i_, j_] := Binomial[f * j - i + d, f * j + a];
mata[i_, j_] := mata1[i, j] + mata2[i, j];
mata[i_Integer, j_Integer] := FunctionExpand[mata1[i, j] + mata2[i, j]];

prodform[0] = 1;
SetDelayed @@
  (Hold[prodform[n_], If[IntegerQ[n], FunctionExpand[C /. prod → Product], C]] /.
    {C → 2 * prod[(Gamma[6 i - 1] Gamma[(i+3)/4]) / (Gamma[5 i] Gamma[(5 i-1)/4]), {i, 1, n}]});

Print[">>> With the following choice of parameters:\n",
  "{a, b, c, d} = ", {a, b, c, d}, ";\n", "{l, m} = ",
  {e, f}, ";\n\nWe are going to prove:\n", TraditionalForm[
   HoldForm @@ {Subscript[det, 0 ≤ i, j < n][e^(i + b) Binomial[f * j + i + c, f * j + a] +
        Binomial[f * j - i + d, f * j + a]] == prodform[n] /. prod → Product}], "\n"];

Print["The matrix of ", Subscript["a", "i,j"], " begins with:\n",
  TableForm[Table[mata[i, j], {i, 0, 5}, {j, 0, 5}]], "\n"];

Print["The determinants begin with:\n",
  Table[Det[Table[mata[i, j], {i, 0, n - 1}, {j, 0, n - 1}]], {n, 1, 6}], "\n"];

Print["The product formula begins with:\n", Table[prodform[n], {n, 1, 6}]];
```

We are going to evaluate the determinant:

$$\det_{0\le i,j<n}\left(\binom{d-i+f j}{a+f j}+e^{b+i}\binom{c+i+f j}{a+f j}\right)$$

```
>>> With the following choice of parameters:
{a, b, c, d} = {3, 0, 3, 3};
{l, m} = {2, 4};

We are going to prove:
```

$$\det_{0\le i,j<n}\left(\binom{3-i+4j}{3+4j}+2^{i}\binom{3+i+4j}{3+4j}\right)=2\prod_{i=1}^{n}\frac{\Gamma\left(\frac{3+i}{4}\right)\Gamma(-1+6i)}{\Gamma(5i)\,\Gamma\left(\frac{1}{4}(-1+5i)\right)}$$

The matrix of $a_{i,j}$ begins with:

2	2	2	2	2	2
8	16	24	32	40	48
40	144	312	544	840	1200
160	960	2912	6528	12 320	20 800
559	5280	21 840	62 016	141 680	280 800
1788	25 344	139 776	496 128	1 360 128	3 144 960

```
The determinants begin with:
{2, 16, 1024, 524 288, 2 146 959 360, 70 300 024 700 928}

The product formula begins with:
{2, 16, 1024, 524 288, 2 146 959 360, 70 300 024 700 928}
```

Construct the minor-related quantity $c_{n,j}$.

We will generate the data of $c_{n,j}$ in advance. **No need to execute the following codes again.** Instead, import the data directly.

```
In[]:= start = CurrentDate[];

ClearAll[DATAC, MATC];

MAX = 70;

DATAC[n_Integer] := DATAC[n] =
   With[{ns = NullSpace[Table[mata[i, j], {i, 0, n - 2}, {j, 0, n - 1}]][[1]]},
    Together[ns / Last[ns]]];
MATC[n_Integer, j_Integer] := Which[n == 1 && j == 0, 1, j ≥ n, 0, True, DATAC[n][[j + 1]]];

Export["datac.txt", {Table[MATC[n, j], {n, MAX}, {j, 0, n - 1}]}]

Print["Time used: ", CurrentDate[] - start];

Out[]= datac.txt

Time used: 16.8852 s
```

Import the data of $c_{n,j}$.

```
In[]:= DATACImported = ToExpression[Import["datac.txt"]];

datac[n_Integer] := datac[n] = DATACImported[[n]];
matc[n_, j_] := matc[n, j] = Piecewise[{{datac[n][[j + 1]], j < n}}, 0];

Print["The matrix of ", Subscript["c", "n,j"],
  " begins with:\n", TableForm[Table[matc[n, j], {n, 1, 6}, {j, 0, n - 1}]]];
```

The matrix of $c_{n,j}$ begins with:

1					
-1	1				
1	-2	1			
-1	3	-3	1		
1	-4	6	-4	1	
$-\frac{4096}{4095}$	$\frac{20479}{4095}$	$-\frac{13652}{1365}$	$\frac{40954}{4095}$	$-\frac{20476}{4095}$	1

Guess the annihilator for $c_{n,j}$.

We will generate the guessed annihilator for $c_{n,j}$ in advance. **No need to execute the following codes again.** Instead, import the data directly.

```
In[]:= start = CurrentDate[];

MAX = 60;

ClearAll[cc, n, j];

guess =
  GuessMultRE[Table[Piecewise[{{matc[n, j], j ≤ n - 1}}, 0], {n, 1, MAX}, {j, 0, MAX - 1}],
   Flatten[Table[cc[n + l1, j + l2], {l1, 0, 3}, {l2, 0, 4}]],
   {n, j}, 8, StartPoint → {1, 0}, Constraints → (j < n)];

Print["Time used: ", CurrentDate[] - start];
```

Time used: 1.22902 min

```
In[]:= start = CurrentDate[];

annc = OreGroebnerBasis[NormalizeCoefficients /@ ToOrePolynomial[guess, cc[n, j]]];
AnnInfo[annc]
Export["annc.txt", {annc}]

Print["Time used: ", CurrentDate[] - start];
```

ByteCount: 442 080

Support: $\{\{S_n^2, S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n S_j^2, S_n S_j, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{15, 15}, {9, 13}, {6, 6}}

Standard Monomials: $\{1, S_j, S_n, S_j^2, S_n S_j\}$

Holonomic Rank: 5

Out[]= annc.txt

Time used: 7.20385 min

Import the annihilator for $c_{n,j}$.

In[•]:=
```
ClearAll[n, j, cc];

annc = ToExpression[Import["annc.txt"]];

AnnInfo[annc]

Print[];

MAX = 6;
Print["Check whether the first values of ",
  Subscript["c", "n,j"], " satisfy the guessed recurrences:\n",
  Union[Flatten[Table[Together[ApplyOreOperator[annc, cc[n, j]] /.
      {n → nn, j → jj, cc → matc}], {nn, 1, MAX}, {jj, 0, nn - 1}]]]];

Print[];

Print["The values at these indices have to be given as initial conditions,
    in order to uniquely define ",
  Subscript["c", "n,j"], " via the recurrences in annc:\n",
  AnnihilatorSingularities[annc, First /@ OreAlgebra[annc][[1]] /. {n → 1, j → 0}]];
```

ByteCount: 441 504

Support: $\{\{S_n^2, S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n S_j^2, S_n S_j, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{15, 15}, {9, 13}, {6, 6}}

Standard Monomials: $\{1, S_j, S_n, S_j^2, S_n S_j\}$

Holonomic Rank: 5

Check whether the first values of $c_{n,j}$ satisfy the guessed recurrences:

{0}

The values at these indices have to be given as initial conditions,
  in order to uniquely define $c_{n,j}$ via the recurrences in annc:

{{{j → 0, n → 1}, True}, {{j → 0, n → 2}, True},
  {{j → 1, n → 1}, True}, {{j → 1, n → 2}, True}, {{j → 2, n → 1}, True}}

Proof of (H1)

Compute a recurrence for $c_{n,n-1}$.

In[•]:=
```
start = CurrentDate[];

ClearAll[n, j];
Support[cnn1 = DFiniteSubstitute[annc, {j → n - 1}][[1]]]

Print["Time used: ", CurrentDate[] - start];
```

Out[•]= $\{S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}$

Time used: 37.0925 s

Verify that this recurrence admits a constant sequence as solution.

In[•]:=
```
OreReduce1[cnn1, Annihilator[1, S[n]]]
```

Out[•]= 0

Look at the integer roots of the leading coefficient.

```
In[∘]:= Select[n /. Solve[LeadingCoefficient[cnn1] == 0, n], IntegerQ]
Out[∘]= {-5}
```

Check the first few initial values.

```
In[∘]:= Table[matc[n, n - 1], {n, 9}]
Out[∘]= {1, 1, 1, 1, 1, 1, 1, 1, 1}
```

Proof of (H2)

Include the variable i into annc.

```
In[∘]:= ClearAll[n, j, i];

annci = ToOrePolynomial[Prepend[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
```

Annihilator for $a_{i,j} * c_{n,j}$. Recall that $a_{i,j}$ is split into two parts $a1_{i,j} + a2_{i,j}$. **No need to execute the following codes again.** Instead, import the data directly.

```
In[∘]:= start = CurrentDate[];

annH2Smnd1 = DFiniteTimesHyper[annci, mata1[i, j]];
annH2Smnd2 = DFiniteTimesHyper[annci, mata2[i, j]];
Export["annH2Smnd1.txt", {annH2Smnd1}]
Export["annH2Smnd2.txt", {annH2Smnd2}]

Print["Time used: ", CurrentDate[] - start];
Out[∘]= annH2Smnd1.txt
Out[∘]= annH2Smnd2.txt

Time used: 24.1555 s
```

$a1_{i,j} * c_{n,j}$

Import the annihilator for $a1_{i,j} * c_{n,j}$.

```
In[∘]:= annH2Smnd1 = ToExpression[Import["annH2Smnd1.txt"]];
AnnInfo[annH2Smnd1]
```

ByteCount: 6 192 232
Support:
$\{\{S_i, 1\}, \{S_n^2, S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n S_j^2, S_n S_j, S_j^2, S_n, S_j, 1\}\}$
degree {n, j, i}: {{0, 1, 1}, {15, 23, 8}, {9, 25, 12}, {6, 14, 8}}
Standard Monomials: $\{1, S_j, S_n, S_j^2, S_n S_j\}$
Holonomic Rank: 5

Import the 1st telescoper for $a1_{i,j} * c_{n,j}$.

```
In[∘]:= annH2CT1No1 = ToExpression[Import["annH2CT1No1.txt"]];

In[∘]:= AnnInfo[annH2CT1No1]
```

```
ByteCount: 39 571 320
Support: {{S_i^5, S_i^4 S_n, S_i^3 S_n^2, S_i^2 S_n^3, S_i S_n^4, S_n^5,
  S_i^4, S_i^3 S_n, S_i^2 S_n^2, S_i S_n^3, S_n^4, S_i^3, S_i^2 S_n, S_i S_n^2, S_n^3, S_i^2, S_i S_n, S_n^2, S_i, S_n, 1}}
degree {i, n}: {{87, 85}}
Standard Monomials: ∞
Holonomic Rank: 1
```

In[∘]:=
```
deltaH2CT1No1 = ToExpression[Import["deltaH2CT1No1.txt"]];
```

In[∘]:=
```
ByteCount[deltaH2CT1No1]
```

Out[∘]=
```
3 876 381 000
```

Verify the 1st telescoper for $a1_{i,j} * c_{n,j}$. **Note that this step is VERY Memory-consuming, so we executed the command on an Amazon AWS cluster.** To illustrate the process, we may nonrigorously substitute the parameters with specific values.

In[∘]:=
```
Timing[OreReduce[
  MapThread[(#1 + (S[j] - 1) ** #2) &, {annH2CT1No1, deltaH2CT1No1}], annH2Smnd1]]
```

Out[∘]=
```
$Aborted
```

In[∘]:=
```
subs = {n → 23, i → 135};
{annH2CT1No1subs, deltaH2CT1No1subs} =
  OrePolynomialSubstitute[#, subs] & /@ {annH2CT1No1, deltaH2CT1No1};
Timing[OreReduce[MapThread[(#1 + (S[j] - 1) ** #2) &,
   {annH2CT1No1subs, deltaH2CT1No1subs}], annH2Smnd1, OrePolynomialSubstitute → subs]]
```

Out[∘]=
```
{33.5, {0}}
```

In[∘]:=
```
subs = {n → 511, i → 100};
{annH2CT1No1subs, deltaH2CT1No1subs} =
  OrePolynomialSubstitute[#, subs] & /@ {annH2CT1No1, deltaH2CT1No1};
Timing[OreReduce[MapThread[(#1 + (S[j] - 1) ** #2) &,
   {annH2CT1No1subs, deltaH2CT1No1subs}], annH2Smnd1, OrePolynomialSubstitute → subs]]
```

Out[∘]=
```
{38.25, {0}}
```

Import the 2nd telescoper for $a1_{i,j} * c_{n,j}$.

In[∘]:=
```
annH2CT1No2 = ToExpression[Import["annH2CT1No2.txt"]];
```

In[∘]:=
```
AnnInfo[annH2CT1No2]
```

```
ByteCount: 31 668 024
Support: {{S_i^6, S_i^5, S_i^4 S_n, S_i^3 S_n^2, S_i^2 S_n^3, S_i S_n^4,
  S_i^4, S_i^3 S_n, S_i^2 S_n^2, S_i S_n^3, S_n^4, S_i^3, S_i^2 S_n, S_i S_n^2, S_n^3, S_i^2, S_i S_n, S_n^2, S_i, S_n, 1}}
degree {i, n}: {{81, 72}}
Standard Monomials: ∞
Holonomic Rank: 1
```

In[∘]:=
```
deltaH2CT1No2 = ToExpression[Import["deltaH2CT1No2.txt"]];
```

In[∘]:=
```
ByteCount[deltaH2CT1No2]
```

Out[∘]=
```
2 413 087 160
```

Verify the 2nd telescoper for $a1_{i,j} * c_{n,j}$. **Note that this step is VERY Memory-consuming, so we executed the command on an Amazon AWS cluster.** To illustrate the process, we may nonrigorously substitute the parameters with specific values.

In[∘]:=
```
Timing[OreReduce[
  MapThread[(#1 + (S[j] - 1) ** #2) &, {annH2CT1No2, deltaH2CT1No2}], annH2Smnd1]]
```
Out[∘]=
```
$Aborted
```

In[∘]:=
```
subs = {n → 23, i → 135};
{annH2CT1No2subs, deltaH2CT1No2subs} =
  OrePolynomialSubstitute[#, subs] & /@ {annH2CT1No2, deltaH2CT1No2};
Timing[OreReduce[MapThread[(#1 + (S[j] - 1) ** #2) &,
   {annH2CT1No2subs, deltaH2CT1No2subs}], annH2Smnd1, OrePolynomialSubstitute → subs]]
```
Out[∘]=
```
{24.375, {0}}
```

In[∘]:=
```
subs = {n → 511, i → 100};
{annH2CT1No2subs, deltaH2CT1No2subs} =
  OrePolynomialSubstitute[#, subs] & /@ {annH2CT1No2, deltaH2CT1No2};
Timing[OreReduce[MapThread[(#1 + (S[j] - 1) ** #2) &,
   {annH2CT1No2subs, deltaH2CT1No2subs}], annH2Smnd1, OrePolynomialSubstitute → subs]]
```
Out[∘]=
```
{22.7031, {0}}
```

$a2_{i,j} * c_{n,j}$

Import the annihilator for $a2_{i,j} * c_{n,j}$.

In[∘]:=
```
annH2Smnd2 = ToExpression[Import["annH2Smnd2.txt"]];
AnnInfo[annH2Smnd2]
```

ByteCount: 6 189 360

Support:
$\{\{S_i, 1\}, \{S_n^2, S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n S_j^2, S_n S_j, S_j^2, S_n, S_j, 1\}\}$

degree {n, j, i}: {{0, 1, 1}, {15, 23, 8}, {9, 25, 12}, {6, 14, 8}}

Standard Monomials: $\{1, S_j, S_n, S_j^2, S_n S_j\}$

Holonomic Rank: 5

Import the 1st telescoper for $a2_{i,j} * c_{n,j}$.

In[∘]:=
```
annH2CT2No1 = ToExpression[Import["annH2CT2No1.txt"]];
```

In[∘]:=
```
AnnInfo[annH2CT2No1]
```

ByteCount: 36 878 968

Support: $\{\{S_i^5, S_i^4 S_n, S_i^3 S_n^2, S_i^2 S_n^3, S_i S_n^4, S_n^5,$
$S_i^4, S_i^3 S_n, S_i^2 S_n^2, S_i S_n^3, S_n^4, S_i^3, S_i^2 S_n, S_i S_n^2, S_n^3, S_i^2, S_i S_n, S_n^2, S_i, S_n, 1\}\}$

degree {i, n}: {{82, 85}}

Standard Monomials: ∞

Holonomic Rank: 1

In[∘]:=
```
deltaH2CT2No1 = ToExpression[Import["deltaH2CT2No1.txt"]];
```

In[∘]:=
```
ByteCount[deltaH2CT2No1]
```
Out[∘]=
```
5 048 293 960
```

Verify the 1st telescoper for $a2_{i,j} * c_{n,j}$. **Note that this step is VERY Memory-consuming, so we executed the command on an Amazon AWS cluster.** To illustrate the process, we may nonrigorously substitute the parameters with specific values.

In[∘]:=
```
Timing[OreReduce[
  MapThread[(#1 + (S[j] - 1) ** #2) &, {annH2CT2No1, deltaH2CT2No1}], annH2Smnd2]]
```

Out[•]= $Aborted

```
In[•]:= subs = {n → 23, i → 135};
{annH2CT2No1subs, deltaH2CT2No1subs} =
  OrePolynomialSubstitute[#, subs] & /@ {annH2CT2No1, deltaH2CT2No1};
Timing[OreReduce[MapThread[(#1 + (S[j] - 1) ** #2) &,
   {annH2CT2No1subs, deltaH2CT2No1subs}], annH2Smnd2, OrePolynomialSubstitute → subs]]
```

Out[•]= {35.5781, {0}}

```
In[•]:= subs = {n → 511, i → 100};
{annH2CT2No1subs, deltaH2CT2No1subs} =
  OrePolynomialSubstitute[#, subs] & /@ {annH2CT2No1, deltaH2CT2No1};
Timing[OreReduce[MapThread[(#1 + (S[j] - 1) ** #2) &,
   {annH2CT2No1subs, deltaH2CT2No1subs}], annH2Smnd2, OrePolynomialSubstitute → subs]]
```

Out[•]= {33.2656, {0}}

Import the 2nd telescoper for a2$_{i,j}$ * $c_{n,j}$.

```
In[•]:= annH2CT2No2 = ToExpression[Import["annH2CT2No2.txt"]];

In[•]:= AnnInfo[annH2CT2No2]
```

ByteCount: 28 931 264

Support: $\{\{S_i^6, S_i^5, S_i^4 S_n, S_i^3 S_n^2, S_i^2 S_n^3, S_i S_n^4, S_i^4, S_i^3 S_n, S_i^2 S_n^2, S_i S_n^3, S_n^4, S_i^3, S_i^2 S_n, S_i S_n^2, S_n^3, S_i^2, S_i S_n, S_n^2, S_i, S_n, 1\}\}$

degree {i, n}: {{75, 72}}

Standard Monomials: ∞

Holonomic Rank: 1

```
In[•]:= deltaH2CT2No2 = ToExpression[Import["deltaH2CT2No2.txt"]];

In[•]:= ByteCount[deltaH2CT2No2]
```

Out[•]= 3 177 278 424

Verify the 2nd telescoper for a2$_{i,j}$ * $c_{n,j}$. **Note that this step is VERY Memory-consuming, so we executed the command on an Amazon AWS cluster.** To illustrate the process, we may nonrigorously substitute the parameters with specific values.

```
In[•]:= Timing[OreReduce[
  MapThread[(#1 + (S[j] - 1) ** #2) &, {annH2CT2No2, deltaH2CT2No2}], annH2Smnd2]]
```

Out[•]= $Aborted

```
In[•]:= subs = {n → 23, i → 135};
{annH2CT2No2subs, deltaH2CT2No2subs} =
  OrePolynomialSubstitute[#, subs] & /@ {annH2CT2No2, deltaH2CT2No2};
Timing[OreReduce[MapThread[(#1 + (S[j] - 1) ** #2) &,
   {annH2CT2No2subs, deltaH2CT2No2subs}], annH2Smnd2, OrePolynomialSubstitute → subs]]
```

Out[•]= {22.5156, {0}}

```
In[•]:= subs = {n → 511, i → 100};
{annH2CT2No2subs, deltaH2CT2No2subs} =
  OrePolynomialSubstitute[#, subs] & /@ {annH2CT2No2, deltaH2CT2No2};
Timing[OreReduce[MapThread[(#1 + (S[j] - 1) ** #2) &,
   {annH2CT2No2subs, deltaH2CT2No2subs}], annH2Smnd2, OrePolynomialSubstitute → subs]]
```

Out[∘]= {23.1875, {0}}

$a_{i,j} * c_{n,j}$

Check that annH2CT1No1 and annH2CT2No1 differ by a scalar. So annH2CT1No1 also annihilates $\sum_j a2_{i,j} * c_{n,j}$.

```
In[∘]:= annH2CT1No1LeadingCoefficient = LeadingCoefficient[annH2CT1No1[[1]]];
annH2CT2No1LeadingCoefficient = LeadingCoefficient[annH2CT2No1[[1]]];
ScalarNo1 = Factor[annH2CT1No1LeadingCoefficient / annH2CT2No1LeadingCoefficient]
Out[∘]= (1 + i) (2 + i) (3 + i) (4 + i) (5 + i)

In[∘]:= annH2CT1No1CoeffList = OrePolynomialListCoefficients[annH2CT1No1[[1]]];
annH2CT2No1CoeffList = OrePolynomialListCoefficients[annH2CT2No1[[1]]];

In[∘]:= Expand[annH2CT1No1CoeffList] == Expand[ScalarNo1 * annH2CT2No1CoeffList]
Out[∘]= True
```

Check that annH2CT1No2 and annH2CT2No2 differ by a scalar. So annH2CT1No2 also annihilates $\sum_j a2_{i,j} * c_{n,j}$.

```
In[∘]:= annH2CT1No2LeadingCoefficient = LeadingCoefficient[annH2CT1No2[[1]]];
annH2CT2No2LeadingCoefficient = LeadingCoefficient[annH2CT2No2[[1]]];
ScalarNo2 = Factor[annH2CT1No2LeadingCoefficient / annH2CT2No2LeadingCoefficient]
Out[∘]= (1 + i) (2 + i) (3 + i) (4 + i) (5 + i) (6 + i)

In[∘]:= annH2CT1No2CoeffList = OrePolynomialListCoefficients[annH2CT1No2[[1]]];
annH2CT2No2CoeffList = OrePolynomialListCoefficients[annH2CT2No2[[1]]];

In[∘]:= Expand[annH2CT1No2CoeffList] == Expand[ScalarNo2 * annH2CT2No2CoeffList]
Out[∘]= True

In[∘]:= annH2CTNo1 = annH2CT1No1;
annH2CTNo2 = annH2CT1No2;
```

Determine the singularities with $i \geq 5$.

```
In[∘]:= LeadingExponent[annH2CTNo1[[1]]]
Out[∘]= {5, 0}

In[∘]:= LeadingExponent[annH2CTNo2[[1]]]
Out[∘]= {6, 0}

In[∘]:= coeff1 = LeadingCoefficient[annH2CTNo1[[1]]] /.
   {i → i - LeadingExponent[annH2CTNo1[[1]]][[1]],
    n → n - LeadingExponent[annH2CTNo1[[1]]][[2]]};

coeff2 = LeadingCoefficient[annH2CTNo2[[1]]] /.
   {i → i - LeadingExponent[annH2CTNo2[[1]]][[1]],
    n → n - LeadingExponent[annH2CTNo2[[1]]][[2]]};
```

Check the singularities at $i = 5$.

```
In[∘]:= Solve[(coeff1 /. {i → 5}) == 0 && 5 < n - 1, n, Integers]
Out[∘]= {}
```

Check the singularities at $i \geq 6$.

The following code is too Time- and especially Memory-consuming so we finally executed it on an Amazon cluster. The output is empty { }. (To convince the reader, we also perform some numerical verification.)

```
In[]:= Solve[coeff1 == 0 && coeff2 == 0 && i ≥ 6 && i < n - 1, {n, i}, Integers]
Out[]= $Aborted
```

```
In[]:= sol = {};
For[ii = 6, ii ≤ 100, ii++,
  For[nn = ii + 2, nn ≤ ii + 100, nn++,
   If[(coeff1 /. {n → nn, i → ii}) == 0 && (coeff2 /. {n → nn, i → ii}) == 0,
    AppendTo[sol, {n → nn, i → ii}];
    ];
   ];
  ];
sol
Out[]= {}
```

Check values at $i = 0$.

```
In[]:= ii = 0;
```

Apply creative telescoping to $a1_{i,j} * c_{n,j}$.

```
In[]:= annii = DFiniteTimesHyper[annc, mata1[ii, j]];
{anniiCT, deltaiiCT} = FindCreativeTelescoping[annii, S[j] - 1];
```

The telescoper part is 1. We then find that the inhomogeneous part is **nonvanishing**, and hence the analysis becomes trickier.

```
In[]:= anniiCT
Out[]= {1}
```

Here we note that the delta part is supported on a collection of power products $S[n]^a\ S[j]^b$ with, in particular, $0 \leq b \leq 2$.

```
In[]:= Support[deltaiiCT[[1]]]
Out[]= {{Sn Sj, Sj^2, Sn, Sj, 1}}
```

```
In[]:= deltaiiCT[[1, 1]][[1, All, 2]]
BB = Max[%[[All, 2]]]
Out[]= {{1, 1}, {0, 2}, {1, 0}, {0, 1}, {0, 0}}
Out[]= 2
```

We write explicitly the inhomogeneous part by calling $c_{n,j}$ temporarily under the name ff[n,j].

```
In[]:= ClearAll[ff];
inhomii = ApplyOreOperator[deltaiiCT[[1]], mata1[ii, j] * ff[n, j]][[1]];
```

The inhomogeneous part at $j = n$ vanishes by recalling that $c_{n,n-1} = 1$ and $c_{n,j} = 0$ when $j \geq n$.

```
In[]:= sumiin = (inhomii /. {j → n}) /.
   {(ff[a_, b_] /; (SameQ[a - b, 1])) → 1, (ff[a_, b_] /; (a - b ≤ 0)) → 0}
```

Out[∘]= 0

The inhomogeneous part at $j = 0$ will be split into parts by collecting $c_{?,\mathrm{b}}$.

```
In[∘]:= sumii = inhomii /. {j → 0};
```

```
In[∘]:= ClearAll[part];
For[bb = 0, bb ≤ BB, bb++,
  part[bb] = sumii /. {(ff[a_, b_] /; SameQ[b, bb] == False) → 0};
 ];
```

We shall see that by combining these parts, our $\Sigma^{(1)}_{i,n}$ will be recovered.

```
In[∘]:= MAX = 3;
For[bb = 0, bb ≤ BB, bb++,
  Print["Part ", bb, ":\n",
   Table[part[bb] /. {n → nn, ff → matc}, {nn, ii + 2, ii + 2 + MAX}], "\n=========="];
 ];
Print["Total:\n", Table[Sum[part[bb] /. {n → nn, ff → matc}, {bb, 0, BB}],
   {nn, ii + 2, ii + 2 + MAX}], "\n====================\n",
  "Compare with ∑j", Subscript["a1", ToString[ii] <> ",j"], "cn,j:\n",
  Table[Sum[mata1[ii, j] * matc[nn, j], {j, 0, nn - 1}], {nn, ii + 2, ii + 2 + MAX}]];
```

Part 0:

$\left\{-\frac{14953}{3328}, -\frac{235575}{53504}, \frac{48968969}{43718400}, -\frac{240257791}{620238080}\right\}$

==========

Part 1:

$\left\{\frac{14953}{3328}, \frac{45357}{6688}, -\frac{37856167}{21859200}, \frac{372750613}{620238080}\right\}$

==========

Part 2:

$\left\{0, -\frac{609}{256}, \frac{783}{1280}, -\frac{957}{4480}\right\}$

==========

Total:

{0, 0, 0, 0}

====================

Compare with $\sum_j \mathrm{a1}_{0,j} c_{n,j}$:

{0, 0, 0, 0}

```
In[∘]:= MAX = 10;
Table[Sum[mata1[ii, j] * matc[nn, j], {j, 0, nn - 1}], {nn, ii + 2, ii + MAX}] ==
 Table[Sum[part[bb] /. {n → nn, ff → matc}, {bb, 0, BB}], {nn, ii + 2, ii + MAX}]
```

Out[∘]= True

Derive separately the annihilating ideal anncii[b] for the univariate sequence $c_{n,j}$ with fixed $j = b$.

```
In[∘]:= ClearAll[anncii];
For[bb = 0, bb ≤ BB, bb++,
  anncii[bb] = DFiniteSubstitute[annc, {j → bb}];
 ];
```

In what follows, we show that $\Sigma^{(1)}_{i,n}$ is annihilated by the annihilating ideal anncii[0] for $c_{n,0}$.

For each part obtained earlier, we apply the Ore polynomial anncii[0].

Putting them together, the Ore polynomial anncii[0] is indeed applied to $\Sigma_{i,n}^{(1)}$; the annihilation is illustrated numerically below.

In[∘]:=
```
ClearAll[partunderannc0];
MAX = 15;
For[bb = 0, bb ≤ BB, bb++,
  partunderannc0[bb] = ApplyOreOperator[anncii[0][[1]], part[bb]];
 ];
Table[Sum[partunderannc0[bb] /. {n → nn, ff → matc}, {bb, 0, BB}], {nn, ii + 2, ii + MAX}]
```

Out[∘]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Note that the b-th part acted by anncii[0] can be rewritten as the univariate sequence $c_{n,b}$ acted by a certain Ore polynomial O_b in the algebra $S[n]$. Now we derive an annihilating ideal for each $O_b \cdot c_{n,b}$.

In[∘]:=
```
ClearAll[orepolypart, annpart];
For[bb = 0, bb ≤ BB, bb++,
  orepolypart[bb] = ToOrePolynomial[partunderannc0[bb], ff[n, bb]];
  annpart[bb] = DFiniteOreAction[anncii[bb], orepolypart[bb]];
 ];
```

We can indeed check the correctness of these annihilating ideals numerically.

In[∘]:=
```
ClearAll[gg];
ClearAll[recpart, valpart];
MAX = 15;
For[bb = 0, bb ≤ BB, bb++,
  recpart[bb] = ApplyOreOperator[annpart[bb], gg[n]][[1]];
  valpart[bb][nn_] := partunderannc0[bb] /. {n → nn, ff → matc};
  Print["Part ", bb, ":\n", Table[
    recpart[bb] /. {n → nn, gg → valpart[bb]}, {nn, ii + 2, ii + MAX}], "\n=========="];
 ];
```

```
Part 0:
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
==========
Part 1:
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
==========
Part 2:
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
==========
```

By the closure property, we obtain an annihilating ideal for $\sum_b O_b \cdot c_{n,b} = \text{anncii}[0] \cdot \Sigma_{i,n}^{(1)}$. We then compute the singularities.

In[∘]:=
```
anntotal = DFinitePlus[Sequence @@ Table[annpart[bb], {bb, 0, BB}]];
```

In[∘]:=
```
LeadingExponent[anntotal[[1]]][[1]]
```

Out[∘]=
```
13
```

In[∘]:=
```
Solve[
 (LeadingCoefficient[anntotal[[1]]] /. {n → n - LeadingExponent[anntotal[[1]]][[1]]}) ==
  0, n, Integers]
```

Out[∘]=
```
{{n → -8}, {n → -7}, {n → -7}, {n → -5}, {n → -3}, {n → -2}, {n → -1}, {n → 0}, {n → 1}}
```

We check that the initial values for $\sum_b O_b \cdot c_{n,b} = \text{anncii}[0] \cdot \Sigma_{i,n}^{(1)}$ are all 0, and hence they vanish for all

$n \geq i+2$. **This confirms that $\Sigma_{i,n}^{(1)}$ is annihilated by anncii[0].**

```
In[•]:= Table[Sum[partunderannc0[bb] /. {n → nn, ff → matc}, {bb, 0, BB}],
  {nn, ii + 2, ii + 2 + LeadingExponent[anntotal[[1]]][[1]]}]

ClearAll[sigma1, hh];
sigma[nn_] := Sum[mata1[ii, j] * matc[nn, j], {j, 0, nn - 1}];
sigma1underannc0 = ApplyOreOperator[anncii[0][[1]], hh[n]];
Table[sigma1underannc0 /. {n → nn, hh → sigma},
  {nn, ii + 2, ii + 2 + LeadingExponent[anntotal[[1]]][[1]]}]
Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Note that $a2_{i,j}$ equals Binomial[$-i+3$, 3] when $j=0$, and 0 when $j \geq 1$ for our choice of i. **Hence, $\Sigma_{i,n}^{(2)} = \sum_{j=0}^{n-1} a2_{i,j}\, c_{n,j} =$ Binomial[$-i+3$, 3]$* c_{n,0}$ is also annihilated by anncii[0].**

Consequently, $\Sigma_{i,n}$ is annihilated by anncii[0].

Finally, after computing the singularities, we check that $\Sigma_{i,n}$ vanishes for its initial values and hence for all $n \geq i+2$.

```
In[•]:= LeadingExponent[anncii[0][[1]]][[1]]
Out[•]= 5

In[•]:= Solve[(LeadingCoefficient[anncii[0][[1]]] /.
      {n → n - LeadingExponent[anncii[0][[1]]][[1]]}) == 0, n, Integers]
Out[•]= {{n → -1}, {n → 1}}

In[•]:= Table[Sum[mata[ii, j] * matc[n, j], {j, 0, n - 1}],
  {n, ii + 2, ii + 2 + LeadingExponent[anncii[0][[1]]][[1]]}]
Out[•]= {0, 0, 0, 0, 0, 0}
```

Check values at $i = 1$.

```
In[•]:= ii = 1;
```

Apply creative telescoping to $a1_{i,j} * c_{n,j}$.

```
In[•]:= annii = DFiniteTimesHyper[annc, mata1[ii, j]];
{anniiCT, deltaiiCT} = FindCreativeTelescoping[annii, S[j] - 1];
```

The telescoper part is 1. We then find that the inhomogeneous part is **nonvanishing**, and hence the analysis becomes trickier.

```
In[•]:= anniiCT
Out[•]= {1}
```

Here we note that the delta part is supported on a collection of power products $S[n]^a\, S[j]^b$ with, in particular, $0 \leq b \leq 2$.

```
In[•]:= Support[deltaiiCT[[1]]]
Out[•]= {{Sn Sj, Sj^2, Sn, Sj, 1}}

In[•]:= deltaiiCT[[1, 1]][[1, All, 2]]
BB = Max[%[[All, 2]]]
```

Out[∘]= {{1, 1}, {0, 2}, {1, 0}, {0, 1}, {0, 0}}

Out[∘]= 2

We write explicitly the inhomogeneous part by calling $c_{n,j}$ temporarily under the name ff[n,j].

```
In[∘]:= ClearAll[ff];
inhomii = ApplyOreOperator[deltaiiCT[[1]], mata1[ii, j] * ff[n, j]][[1]];
```

The inhomogeneous part at $j = n$ vanishes by recalling that $c_{n,n-1} = 1$ and $c_{n,j} = 0$ when $j \geq n$.

```
In[∘]:= sumiin = (inhomii /. {j → n}) /.
   {(ff[a_, b_] /; (SameQ[a - b, 1])) → 1, (ff[a_, b_] /; (a - b ≤ 0)) → 0}
```

Out[∘]= 0

The inhomogeneous part at $j = 0$ will be split into parts by collecting $c_{?,b}$.

```
In[∘]:= sumii = inhomii /. {j → 0};
```

```
In[∘]:= ClearAll[part];
For[bb = 0, bb ≤ BB, bb++,
   part[bb] = sumii /. {(ff[a_, b_] /; SameQ[b, bb] == False) → 0};
  ];
```

We shall see that by combining these parts, our $\Sigma^{(1)}_{i,n}$ will be recovered.

```
In[∘]:= MAX = 3;
For[bb = 0, bb ≤ BB, bb++,
   Print["Part ", bb, ":\n",
     Table[part[bb] /. {n → nn, ff → matc}, {nn, ii + 2, ii + 2 + MAX}], "\n=========="];
  ];
Print["Total:\n", Table[Sum[part[bb] /. {n → nn, ff → matc}, {bb, 0, BB}],
    {nn, ii + 2, ii + 2 + MAX}], "\n====================\n",
   "Compare with ∑_j", Subscript["a1", ToString[ii] <> ",j"], "c_{n,j}:\n",
   Table[Sum[mata1[ii, j] * matc[nn, j], {j, 0, nn - 1}], {nn, ii + 2, ii + 2 + MAX}]];
```

Part 0:

$\left\{-\frac{44275}{152}, \frac{1689323}{24840}, -\frac{690125}{21576}, \frac{14729984}{799533}\right\}$

==========

Part 1:

$\left\{\frac{7850}{19}, -\frac{541097}{6210}, \frac{803399}{21576}, -\frac{832256693}{41575716}\right\}$

==========

Part 2:

$\left\{-\frac{975}{8}, \frac{153}{8}, -\frac{21}{4}, \frac{85325}{53508}\right\}$

==========

Total:

{0, 0, 0, 0}

====================

Compare with $\sum_j a1_{1,j} c_{n,j}$:

{0, 0, 0, 0}

```
In[∘]:= MAX = 10;
Table[Sum[mata1[ii, j] * matc[nn, j], {j, 0, nn - 1}], {nn, ii + 2, ii + MAX}] ==
 Table[Sum[part[bb] /. {n → nn, ff → matc}, {bb, 0, BB}], {nn, ii + 2, ii + MAX}]
```

Out[∘]= True

Derive separately the annihilating ideal anncii[b] for the univariate sequence $c_{n,j}$ with fixed $j = b$.

```
In[∘]:= ClearAll[anncii];
For[bb = 0, bb ≤ BB, bb++,
  anncii[bb] = DFiniteSubstitute[annc, {j → bb}];
 ];
```

In what follows, we show that $\Sigma_{i,n}^{(1)}$ is annihilated by the annihilating ideal anncii[0] for $c_{n,0}$.

For each part obtained earlier, we apply the Ore polynomial anncii[0].

Putting them together, the Ore polynomial anncii[0] is indeed applied to $\Sigma_{i,n}^{(1)}$; the annihilation is illustrated numerically below.

```
In[∘]:= ClearAll[partunderannc0];
MAX = 15;
For[bb = 0, bb ≤ BB, bb++,
  partunderannc0[bb] = ApplyOreOperator[anncii[0][[1]], part[bb]];
 ];
Table[Sum[partunderannc0[bb] /. {n → nn, ff → matc}, {bb, 0, BB}], {nn, ii + 2, ii + MAX}]

Out[∘]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Note that the b-th part acted by anncii[0] can be rewritten as the univariate sequence $c_{n,b}$ acted by a certain Ore polynomial O_b in the algebra $S[n]$. Now we derive an annihilating ideal for each $O_b \cdot c_{n,b}$.

```
In[∘]:= ClearAll[orepolypart, annpart];
For[bb = 0, bb ≤ BB, bb++,
  orepolypart[bb] = ToOrePolynomial[partunderannc0[bb], ff[n, bb]];
  annpart[bb] = DFiniteOreAction[anncii[bb], orepolypart[bb]];
 ];
```

We can indeed check the correctness of these annihilating ideals numerically.

```
In[∘]:= ClearAll[gg];
ClearAll[recpart, valpart];
MAX = 15;
For[bb = 0, bb ≤ BB, bb++,
  recpart[bb] = ApplyOreOperator[annpart[bb], gg[n]][[1]];
  valpart[bb][nn_] := partunderannc0[bb] /. {n → nn, ff → matc};
  Print["Part ", bb, ":\n", Table[
    recpart[bb] /. {n → nn, gg → valpart[bb]}, {nn, ii + 2, ii + MAX}], "\n=========="];
 ];

Part 0:
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
==========

Part 1:
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
==========

Part 2:
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
==========
```

By the closure property, we obtain an annihilating ideal for $\sum_b O_b \cdot c_{n,b} = \text{anncii}[0] \cdot \Sigma_{i,n}^{(1)}$. We then compute the singularities.

In[ ]:=
```
anntotal = DFinitePlus[Sequence @@ Table[annpart[bb], {bb, 0, BB}]];
```

In[ ]:=
```
LeadingExponent[anntotal[[1]]][[1]]
```

Out[ ]=
```
12
```

In[ ]:=
```
Solve[
  (LeadingCoefficient[anntotal[[1]]] /. {n → n - LeadingExponent[anntotal[[1]]][[1]]}) ==
   0, n, Integers]
```

Out[ ]=
```
{{n → -8}, {n → -7}, {n → -5}, {n → -3}, {n → -2}, {n → -1}, {n → 0}, {n → 1}}
```

We check that the initial values for $\sum_b O_b \cdot c_{n,b}$ = anncii[0]$\cdot \Sigma^{(1)}_{i,n}$ are all 0, and hence they vanish for all $n \geq i+2$. **This confirms that $\Sigma^{(1)}_{i,n}$ is annihilated by anncii[0].**

In[ ]:=
```
Table[Sum[partunderannc0[bb] /. {n → nn, ff → matc}, {bb, 0, BB}],
  {nn, ii + 2, ii + 2 + LeadingExponent[anntotal[[1]]][[1]]}]

ClearAll[sigma1, hh];
sigma[nn_] := Sum[mata1[ii, j] * matc[nn, j], {j, 0, nn - 1}];
sigma1underannc0 = ApplyOreOperator[anncii[0][[1]], hh[n]];
Table[sigma1underannc0 /. {n → nn, hh → sigma},
  {nn, ii + 2, ii + 2 + LeadingExponent[anntotal[[1]]][[1]]}]
```

Out[ ]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Out[ ]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Note that $a2_{i,j}$ equals Binomial[$-i+3$, 3] when $j=0$, and 0 when $j \geq 1$ for our choice of i. **Hence, $\Sigma^{(2)}_{i,n} = \sum_{j=0}^{n-1} a2_{i,j}\, c_{n,j}$ = Binomial[$-i+3$, 3]$*c_{n,0}$ is also annihilated by anncii[0].**

Consequently, $\Sigma_{i,n}$ is annihilated by anncii[0].

Finally, after computing the singularities, we check that $\Sigma_{i,n}$ vanishes for its initial values and hence for all $n \geq i+2$.

In[ ]:=
```
LeadingExponent[anncii[0][[1]]][[1]]
```

Out[ ]=
```
5
```

In[ ]:=
```
Solve[(LeadingCoefficient[anncii[0][[1]]] /.
    {n → n - LeadingExponent[anncii[0][[1]]][[1]]}) == 0, n, Integers]
```

Out[ ]=
```
{{n → -1}, {n → 1}}
```

In[ ]:=
```
Table[Sum[mata[ii, j] * matc[n, j], {j, 0, n - 1}],
  {n, ii + 2, ii + 2 + LeadingExponent[anncii[0][[1]]][[1]]}]
```

Out[ ]=
```
{0, 0, 0, 0, 0, 0}
```

Check values at $i = 2$.

In[ ]:=
```
ii = 2;
```

Apply creative telescoping to $a1_{i,j} * c_{n,j}$.

In[ ]:=
```
annii = DFiniteTimesHyper[annc, mata1[ii, j]];
{anniiCT, deltaiiCT} = FindCreativeTelescoping[annii, S[j] - 1];
```

The telescoper part is 1. We then find that the inhomogeneous part is **nonvanishing**, and hence the analysis becomes trickier.

In[∘]:=
```
anniiCT
```
Out[∘]=
```
{1}
```

Here we note that the delta part is supported on a collection of power products $S[n]^a\, S[j]^b$ with, in particular, $0 \le b \le 2$.

In[∘]:=
```
Support[deltaiiCT[[1]]]
```
Out[∘]= $\{\{S_n\, S_j,\, S_j^2,\, S_n,\, S_j,\, 1\}\}$

In[∘]:=
```
deltaiiCT[[1, 1]][[1, All, 2]]
BB = Max[%[[All, 2]]]
```
Out[∘]=
```
{{1, 1}, {0, 2}, {1, 0}, {0, 1}, {0, 0}}
```
Out[∘]=
```
2
```

We write explicitly the inhomogeneous part by calling $c_{n,j}$ temporarily under the name ff[n,j].

In[∘]:=
```
ClearAll[ff];
inhomii = ApplyOreOperator[deltaiiCT[[1]], mata1[ii, j] * ff[n, j]][[1]];
```

The inhomogeneous part at $j = n$ vanishes by recalling that $c_{n,n-1} = 1$ and $c_{n,j} = 0$ when $j \ge n$.

In[∘]:=
```
sumiin = (inhomii /. {j → n}) /.
   {(ff[a_, b_] /; (SameQ[a - b, 1])) → 1, (ff[a_, b_] /; (a - b ≤ 0)) → 0}
```
Out[∘]=
```
0
```

The inhomogeneous part at $j = 0$ will be split into parts by collecting $c_{?,b}$.

In[∘]:=
```
sumii = inhomii /. {j → 0};
```

In[∘]:=
```
ClearAll[part];
For[bb = 0, bb ≤ BB, bb++,
   part[bb] = sumii /. {(ff[a_, b_] /; SameQ[b, bb] == False) → 0};
  ];
```

We shall see that by combining these parts, our $\Sigma^{(1)}_{i,n}$ will be recovered.

In[∘]:=
```
MAX = 3;
For[bb = 0, bb ≤ BB, bb++,
   Print["Part ", bb, ":\n",
     Table[part[bb] /. {n → nn, ff → matc}, {nn, ii + 2, ii + 2 + MAX}], "\n=========="];
  ];
Print["Total:\n", Table[Sum[part[bb] /. {n → nn, ff → matc}, {bb, 0, BB}],
    {nn, ii + 2, ii + 2 + MAX}], "\n====================\n",
   "Compare with ∑_j", Subscript["a1", ToString[ii] <> ",j"], "c_{n,j}:\n",
   Table[Sum[mata1[ii, j] * matc[nn, j], {j, 0, nn - 1}], {nn, ii + 2, ii + 2 + MAX}]];
```

Part 0:

$$\left\{\frac{53\,023\,559}{24\,840},\ -\frac{1\,673\,701}{2436},\ \frac{18\,016\,418\,048}{51\,969\,645},\ -\frac{504\,272\,412\,064}{2\,386\,095\,327}\right\}$$

==========

Part 1:

$$\left\{-\frac{16\,141\,211}{6210},\ \frac{1\,867\,363}{2436},\ -\frac{76\,072\,694\,603}{207\,878\,580},\ \frac{8\,254\,851\,171\,253}{38\,177\,525\,232}\right\}$$

==========

```
Part 2:
{3717/8, -159/2, 5157043/267540, -1305943/267344}
==========

Total:
{0, 0, 0, 0}
====================
Compare with ∑_j a1_{2,j} c_{n,j}:
{0, 0, 0, 0}
```

In[∘]:=
```
MAX = 10;
Table[Sum[mata1[ii, j] * matc[nn, j], {j, 0, nn - 1}], {nn, ii + 2, ii + MAX}] ==
 Table[Sum[part[bb] /. {n → nn, ff → matc}, {bb, 0, BB}], {nn, ii + 2, ii + MAX}]
```

Out[∘]=
```
True
```

Derive separately the annihilating ideal anncii[b] for the univariate sequence $c_{n,j}$ with fixed $j = b$.

In[∘]:=
```
ClearAll[anncii];
For[bb = 0, bb ≤ BB, bb++,
  anncii[bb] = DFiniteSubstitute[annc, {j → bb}];
 ];
```

In what follows, we show that $\Sigma_{i,n}^{(1)}$ is annihilated by the annihilating ideal anncii[0] for $c_{n,0}$.

For each part obtained earlier, we apply the Ore polynomial anncii[0].

Putting them together, the Ore polynomial anncii[0] is indeed applied to $\Sigma_{i,n}^{(1)}$; the annihilation is illustrated numerically below.

In[∘]:=
```
ClearAll[partunderannc0];
MAX = 15;
For[bb = 0, bb ≤ BB, bb++,
  partunderannc0[bb] = ApplyOreOperator[anncii[0][[1]], part[bb]];
 ];
Table[Sum[partunderannc0[bb] /. {n → nn, ff → matc}, {bb, 0, BB}], {nn, ii + 2, ii + MAX}]
```

Out[∘]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Note that the b-th part acted by anncii[0] can be rewritten as the univariate sequence $c_{n,b}$ acted by a certain Ore polynomial O_b in the algebra $S[n]$. Now we derive an annihilating ideal for each $O_b \cdot c_{n,b}$.

In[∘]:=
```
ClearAll[orepolypart, annpart];
For[bb = 0, bb ≤ BB, bb++,
  orepolypart[bb] = ToOrePolynomial[partunderannc0[bb], ff[n, bb]];
  annpart[bb] = DFiniteOreAction[anncii[bb], orepolypart[bb]];
 ];
```

We can indeed check the correctness of these annihilating ideals numerically.

In[ ]:=
```
ClearAll[gg];
ClearAll[recpart, valpart];
MAX = 15;
For[bb = 0, bb ≤ BB, bb++,
  recpart[bb] = ApplyOreOperator[annpart[bb], gg[n]][[1]];
  valpart[bb][nn_] := partunderannc0[bb] /. {n → nn, ff → matc};
  Print["Part ", bb, ":\n", Table[
    recpart[bb] /. {n → nn, gg → valpart[bb]}, {nn, ii + 2, ii + MAX}], "\n=========="];
 ];
```

```
Part 0:
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
==========
Part 1:
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
==========
Part 2:
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
==========
```

By the closure property, we obtain an annihilating ideal for $\sum_b O_b \cdot c_{n,b}$ = anncii[0]$\cdot \Sigma^{(1)}_{i,n}$. We then compute the singularities.

In[ ]:=
```
anntotal = DFinitePlus[Sequence @@ Table[annpart[bb], {bb, 0, BB}]];
```

In[ ]:=
```
LeadingExponent[anntotal[[1]]][[1]]
```

Out[ ]=
```
13
```

In[ ]:=
```
Solve[
 (LeadingCoefficient[anntotal[[1]]] /. {n → n - LeadingExponent[anntotal[[1]]][[1]]}) ==
  0, n, Integers]
```

Out[ ]=
```
{{n → -8}, {n → -7}, {n → -5}, {n → -3}, {n → -2}, {n → -1}, {n → 0}}
```

We check that the initial values for $\sum_b O_b \cdot c_{n,b}$ = anncii[0]$\cdot \Sigma^{(1)}_{i,n}$ are all 0, and hence they vanish for all $n \geq i+2$. **This confirms that $\Sigma^{(1)}_{i,n}$ is annihilated by anncii[0].**

In[ ]:=
```
Table[Sum[partunderannc0[bb] /. {n → nn, ff → matc}, {bb, 0, BB}],
 {nn, ii + 2, ii + 2 + LeadingExponent[anntotal[[1]]][[1]]}]

ClearAll[sigma1, hh];
sigma[nn_] := Sum[mata1[ii, j] * matc[nn, j], {j, 0, nn - 1}];
sigma1underannc0 = ApplyOreOperator[anncii[0][[1]], hh[n]];
Table[sigma1underannc0 /. {n → nn, hh → sigma},
 {nn, ii + 2, ii + 2 + LeadingExponent[anntotal[[1]]][[1]]}]
```

Out[ ]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Out[ ]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Note that $a2_{i,j}$ equals Binomial[$-i+3$, 3] when $j=0$, and 0 when $j \geq 1$ for our choice of i. **Hence, $\Sigma^{(2)}_{i,n} = \sum_{j=0}^{n-1} a2_{i,j}\, c_{n,j}$ = Binomial[$-i+3$, 3]$* c_{n,0}$ is also annihilated by anncii[0].**

Consequently, $\Sigma_{i,n}$ is annihilated by anncii[0].

Finally, after computing the singularities, we check that $\Sigma_{i,n}$ vanishes for its initial values and hence for all $n \geq i+2$.

```
In[•]:= LeadingExponent[anncii[0][[1]]][[1]]
Out[•]= 5

In[•]:= Solve[(LeadingCoefficient[anncii[0][[1]]] /.
          {n → n - LeadingExponent[anncii[0][[1]]][[1]]}) == 0, n, Integers]
Out[•]= {{n → -1}, {n → 1}}

In[•]:= Table[Sum[mata[ii, j] * matc[n, j], {j, 0, n - 1}],
         {n, ii + 2, ii + 2 + LeadingExponent[anncii[0][[1]]][[1]]}]
Out[•]= {0, 0, 0, 0, 0, 0}
```

Check values at $i = 3$.

```
In[•]:= ii = 3;
```

Apply creative telescoping to $a1_{i,j} * c_{n,j}$.

```
In[•]:= annii = DFiniteTimesHyper[annc, mata1[ii, j]];
        {anniiCT, deltaiiCT} = FindCreativeTelescoping[annii, S[j] - 1];
```

The telescoper part is 1. We then find that the inhomogeneous part is **nonvanishing**, and hence the analysis becomes trickier.

```
In[•]:= anniiCT
Out[•]= {1}
```

Here we note that the delta part is supported on a collection of power products $S[n]^a\, S[j]^b$ with, in particular, $0 \le b \le 2$.

```
In[•]:= Support[deltaiiCT[[1]]]
Out[•]= {{Sn Sj, Sj^2, Sn, Sj, 1}}

In[•]:= deltaiiCT[[1, 1]][[1, All, 2]]
        BB = Max[%[[All, 2]]]
Out[•]= {{1, 1}, {0, 2}, {1, 0}, {0, 1}, {0, 0}}
Out[•]= 2
```

We write explicitly the inhomogeneous part by calling $c_{n,j}$ temporarily under the name ff[n,j].

```
In[•]:= ClearAll[ff];
        inhomii = ApplyOreOperator[deltaiiCT[[1]], mata1[ii, j] * ff[n, j]][[1]];
```

The inhomogeneous part at $j = n$ vanishes by recalling that $c_{n,n-1} = 1$ and $c_{n,j} = 0$ when $j \ge n$.

```
In[•]:= sumiin = (inhomii /. {j → n}) /.
          {(ff[a_, b_] /; (SameQ[a - b, 1])) → 1, (ff[a_, b_] /; (a - b ≤ 0)) → 0}
Out[•]= 0
```

The inhomogeneous part at $j = 0$ will be split into parts by collecting $c_{?,b}$.

```
In[•]:= sumii = inhomii /. {j → 0};

In[•]:= ClearAll[part];
        For[bb = 0, bb ≤ BB, bb++,
          part[bb] = sumii /. {(ff[a_, b_] /; SameQ[b, bb] == False) → 0};
         ];
```

We shall see that by combining these parts, our $\Sigma_{i,n}^{(1)}$ will be recovered.

```
In[∘]:= MAX = 3;
For[bb = 0, bb ≤ BB, bb++,
  Print["Part ", bb, ":\n",
   Table[part[bb] /. {n → nn, ff → matc}, {nn, ii + 2, ii + 2 + MAX}], "\n=========="];
 ];
Print["Total:\n", Table[Sum[part[bb] /. {n → nn, ff → matc}, {bb, 0, BB}],
   {nn, ii + 2, ii + 2 + MAX}], "\n====================\n",
  "Compare with ∑j", Subscript["a1", ToString[ii] <> ",j"], "cn,j:\n",
  Table[Sum[mata1[ii, j] * matc[nn, j], {j, 0, nn - 1}], {nn, ii + 2, ii + 2 + MAX}]];
```

Part 0:

$\left\{-\frac{493658231}{37758}, \frac{227690250752}{51969645}, -\frac{5358873364384}{2386095327}, \frac{62144423401472}{44718647589}\right\}$

==========

Part 1:

$\left\{\frac{540893489}{37758}, -\frac{475659588451}{103939290}, \frac{87279106321765}{38177525232}, -\frac{999937419024107}{715498361424}\right\}$

==========

Part 2:

$\left\{-1251, \frac{8699737}{44590}, -\frac{10764007}{267344}, \frac{134361215}{17085712}\right\}$

==========

Total:

{0, 0, 0, 0}

====================

Compare with $\sum_j a1_{3,j} c_{n,j}$:

{0, 0, 0, 0}

```
In[∘]:= MAX = 10;
Table[Sum[mata1[ii, j] * matc[nn, j], {j, 0, nn - 1}], {nn, ii + 2, ii + MAX}] ==
 Table[Sum[part[bb] /. {n → nn, ff → matc}, {bb, 0, BB}], {nn, ii + 2, ii + MAX}]
Out[∘]= True
```

Derive separately the annihilating ideal anncii[b] for the univariate sequence $c_{n,j}$ with fixed $j = b$.

```
In[∘]:= ClearAll[anncii];
For[bb = 0, bb ≤ BB, bb++,
  anncii[bb] = DFiniteSubstitute[annc, {j → bb}];
 ];
```

In what follows, we show that $\Sigma_{i,n}^{(1)}$ is annihilated by the annihilating ideal anncii[0] for $c_{n,0}$.

For each part obtained earlier, we apply the Ore polynomial anncii[0].

Putting them together, the Ore polynomial anncii[0] is indeed applied to $\Sigma_{i,n}^{(1)}$; the annihilation is illustrated numerically below.

```
In[∘]:= ClearAll[partunderannc0];
MAX = 15;
For[bb = 0, bb ≤ BB, bb++,
  partunderannc0[bb] = ApplyOreOperator[anncii[0][[1]], part[bb]];
 ];
Table[Sum[partunderannc0[bb] /. {n → nn, ff → matc}, {bb, 0, BB}], {nn, ii + 2, ii + MAX}]
```

Out[ ]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

Note that the b-th part acted by anncii[0] can be rewritten as the univariate sequence $c_{n,b}$ acted by a certain Ore polynomial O_b in the algebra $S[n]$. Now we derive an annihilating ideal for each $O_b \cdot c_{n,b}$.

```
In[ ]:= ClearAll[orepolypart, annpart];
For[bb = 0, bb ≤ BB, bb++,
  orepolypart[bb] = ToOrePolynomial[partunderannc0[bb], ff[n, bb]];
  annpart[bb] = DFiniteOreAction[anncii[bb], orepolypart[bb]];
 ];
```

We can indeed check the correctness of these annihilating ideals numerically.

```
In[ ]:= ClearAll[gg];
ClearAll[recpart, valpart];
MAX = 15;
For[bb = 0, bb ≤ BB, bb++,
  recpart[bb] = ApplyOreOperator[annpart[bb], gg[n]][[1]];
  valpart[bb][nn_] := partunderannc0[bb] /. {n → nn, ff → matc};
  Print["Part ", bb, ":\n", Table[
    recpart[bb] /. {n → nn, gg → valpart[bb]}, {nn, ii + 2, ii + MAX}], "\n=========="];
 ];
```

```
Part 0:
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
==========
Part 1:
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
==========
Part 2:
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
==========
```

By the closure property, we obtain an annihilating ideal for $\sum_b O_b \cdot c_{n,b}$ = anncii[0] $\cdot \Sigma_{i,n}^{(1)}$. We then compute the singularities.

```
In[ ]:= anntotal = DFinitePlus[Sequence @@ Table[annpart[bb], {bb, 0, BB}]];
```

```
In[ ]:= LeadingExponent[anntotal[[1]]][[1]]
Out[ ]= 13
```

```
In[ ]:= Solve[
  (LeadingCoefficient[anntotal[[1]]] /. {n → n - LeadingExponent[anntotal[[1]]][[1]]}) ==
   0, n, Integers]
Out[ ]= {{n → -8}, {n → -7}, {n → -5}, {n → -3}, {n → -2}, {n → -1}, {n → 0}, {n → 1}}
```

We check that the initial values for $\sum_b O_b \cdot c_{n,b}$ = anncii[0] $\cdot \Sigma_{i,n}^{(1)}$ are all 0, and hence they vanish for all $n \geq i + 2$. **This confirms that $\Sigma_{i,n}^{(1)}$ is annihilated by anncii[0].**

```
In[•]:= Table[Sum[partunderannc0[bb] /. {n → nn, ff → matc}, {bb, 0, BB}],
          {nn, ii + 2, ii + 2 + LeadingExponent[anntotal[[1]]][[1]]}]

        ClearAll[sigma1, hh];
        sigma[nn_] := Sum[mata1[ii, j] * matc[nn, j], {j, 0, nn - 1}];
        sigma1underannc0 = ApplyOreOperator[anncii[0][[1]], hh[n]];
        Table[sigma1underannc0 /. {n → nn, hh → sigma},
          {nn, ii + 2, ii + 2 + LeadingExponent[anntotal[[1]]][[1]]}]
Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Note that $a2_{i,j}$ equals Binomial[$-i+3$, 3] when $j = 0$, and 0 when $j \geq 1$ for our choice of i. **Hence, $\Sigma^{(2)}_{i,n} = \sum_{j=0}^{n-1} a2_{i,j}\, c_{n,j}$ = Binomial[$-i+3$, 3]$* c_{n,0}$ is also annihilated by anncii[0].**

Consequently, $\Sigma_{i,n}$ is annihilated by anncii[0].

Finally, after computing the singularities, we check that $\Sigma_{i,n}$ vanishes for its initial values and hence for all $n \geq i+2$.

```
In[•]:= LeadingExponent[anncii[0][[1]]][[1]]
Out[•]= 5

In[•]:= Solve[(LeadingCoefficient[anncii[0][[1]]] /.
            {n → n - LeadingExponent[anncii[0][[1]]][[1]]}) == 0, n, Integers]
Out[•]= {{n → -1}, {n → 1}}

In[•]:= Table[Sum[mata[ii, j] * matc[n, j], {j, 0, n - 1}],
          {n, ii + 2, ii + 2 + LeadingExponent[anncii[0][[1]]][[1]]}]
Out[•]= {0, 0, 0, 0, 0, 0}
```

Check values at $i = 4$.

```
In[•]:= ii = 4;
```

Apply creative telescoping to $a1_{i,j} * c_{n,j}$.

```
In[•]:= annii = DFiniteTimesHyper[annc, mata1[ii, j]];
        {anniiCT, deltaiiCT} = FindCreativeTelescoping[annii, S[j] - 1];
```

The telescoper part is 1. We then find that the inhomogeneous part is **nonvanishing**, and hence the analysis becomes trickier.

```
In[•]:= anniiCT
Out[•]= {1}
```

Here we note that the delta part is supported on a collection of power products $S[n]^a\, S[j]^b$ with, in particular, $0 \leq b \leq 2$.

```
In[•]:= Support[deltaiiCT[[1]]]
Out[•]= {{Sn Sj, Sj^2, Sn, Sj, 1}}

In[•]:= deltaiiCT[[1, 1]][[1, All, 2]]
        BB = Max[%[[All, 2]]]
Out[•]= {{1, 1}, {0, 2}, {1, 0}, {0, 1}, {0, 0}}
```

Out[∘]= 2

We write explicitly the inhomogeneous part by calling $c_{n,j}$ temporarily under the name ff[n,j].

```
In[∘]:= ClearAll[ff];
inhomii = ApplyOreOperator[deltaiiCT[[1]], mata1[ii, j] * ff[n, j]][[1]];
```

The inhomogeneous part at $j = n$ vanishes by recalling that $c_{n,n-1} = 1$ and $c_{n,j} = 0$ when $j \geq n$.

```
In[∘]:= sumiin = (inhomii /. {j → n}) /.
   {(ff[a_, b_] /; (SameQ[a - b, 1])) → 1, (ff[a_, b_] /; (a - b ≤ 0)) → 0}
Out[∘]= 0
```

The inhomogeneous part at $j = 0$ will be split into parts by collecting $c_{?,b}$.

```
In[∘]:= sumii = inhomii /. {j → 0};
```

```
In[∘]:= ClearAll[part];
For[bb = 0, bb ≤ BB, bb++,
   part[bb] = sumii /. {(ff[a_, b_] /; SameQ[b, bb] == False) → 0};
  ];
```

We shall see that by combining these parts, our $\Sigma^{(1)}_{i,n}$ will be recovered.

```
In[∘]:= MAX = 3;
For[bb = 0, bb ≤ BB, bb++,
   Print["Part ", bb, ":\n",
     Table[part[bb] /. {n → nn, ff → matc}, {nn, ii + 2, ii + 2 + MAX}], "\n=========="];
  ];
Print["Total:\n", Table[Sum[part[bb] /. {n → nn, ff → matc}, {bb, 0, BB}],
    {nn, ii + 2, ii + 2 + MAX}], "\n====================\n",
   "Compare with ∑_j", Subscript["a1", ToString[ii] <> ",j"], "c_n,j:\n",
   Table[Sum[mata1[ii, j] * matc[nn, j], {j, 0, nn - 1}], {nn, ii + 2, ii + 2 + MAX}]];
```

Part 0:

$$\left\{\frac{7\,572\,330\,903\,472}{114\,333\,219}, -\frac{310\,503\,762\,684\,581}{14\,316\,571\,962}, \frac{1\,798\,462\,180\,714\,880}{163\,968\,374\,493}, -\frac{1\,283\,395\,890\,144\,832}{190\,338\,513\,365}\right\}$$

==========

Part 1:

$$\left\{-\frac{2\,556\,407\,989\,663\,621}{36\,586\,630\,080}, \frac{162\,287\,305\,841\,761\,501}{7\,330\,084\,844\,544}, -\frac{463\,326\,943\,481\,152\,847}{41\,975\,903\,870\,208}, \frac{9\,211\,310\,165\,807\,687\,809}{1\,364\,346\,463\,800\,320}\right\}$$

==========

Part 2:

$$\left\{\frac{15\,587\,399\,671}{4\,280\,640}, -\frac{23\,123\,058\,319}{51\,330\,048}, \frac{206\,214\,522\,105}{3\,007\,085\,312}, -\frac{18\,109\,261\,617\,911}{2\,340\,216\,919\,040}\right\}$$

==========

Total:

$$\left\{-\frac{4096}{4095}, \frac{1024}{1023}, -\frac{65\,536}{65\,379}, \frac{65\,536}{65\,231}\right\}$$

====================

Compare with $\sum_j a1_{4,j} c_{n,j}$:

$$\left\{-\frac{4096}{4095}, \frac{1024}{1023}, -\frac{65\,536}{65\,379}, \frac{65\,536}{65\,231}\right\}$$

In[∘]:=
```
MAX = 10;
Table[Sum[mata1[ii, j] * matc[nn, j], {j, 0, nn - 1}], {nn, ii + 2, ii + MAX}] ==
 Table[Sum[part[bb] /. {n → nn, ff → matc}, {bb, 0, BB}], {nn, ii + 2, ii + MAX}]
```
Out[∘]=
```
True
```

Derive separately the annihilating ideal anncii[b] for the univariate sequence $c_{n,j}$ with fixed $j = b$.

In[∘]:=
```
ClearAll[anncii];
For[bb = 0, bb ≤ BB, bb++,
  anncii[bb] = DFiniteSubstitute[annc, {j → bb}];
 ];
```

In what follows, we show that $\Sigma_{i,n}^{(1)}$ is annihilated by the annihilating ideal anncii[0] for $c_{n,0}$.

For each part obtained earlier, we apply the Ore polynomial anncii[0].

Putting them together, the Ore polynomial anncii[0] is indeed applied to $\Sigma_{i,n}^{(1)}$; the annihilation is illustrated numerically below.

In[∘]:=
```
ClearAll[partunderannc0];
MAX = 15;
For[bb = 0, bb ≤ BB, bb++,
  partunderannc0[bb] = ApplyOreOperator[anncii[0][[1]], part[bb]];
 ];
Table[Sum[partunderannc0[bb] /. {n → nn, ff → matc}, {bb, 0, BB}], {nn, ii + 2, ii + MAX}]
```
Out[∘]=
```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Note that the b-th part acted by anncii[0] can be rewritten as the univariate sequence $c_{n,b}$ acted by a certain Ore polynomial O_b in the algebra $S[n]$. Now we derive an annihilating ideal for each $O_b \cdot c_{n,b}$.

In[∘]:=
```
ClearAll[orepolypart, annpart];
For[bb = 0, bb ≤ BB, bb++,
  orepolypart[bb] = ToOrePolynomial[partunderannc0[bb], ff[n, bb]];
  annpart[bb] = DFiniteOreAction[anncii[bb], orepolypart[bb]];
 ];
```

We can indeed check the correctness of these annihilating ideals numerically.

In[∘]:=
```
ClearAll[gg];
ClearAll[recpart, valpart];
MAX = 15;
For[bb = 0, bb ≤ BB, bb++,
  recpart[bb] = ApplyOreOperator[annpart[bb], gg[n]][[1]];
  valpart[bb][nn_] := partunderannc0[bb] /. {n → nn, ff → matc};
  Print["Part ", bb, ":\n", Table[
    recpart[bb] /. {n → nn, gg → valpart[bb]}, {nn, ii + 2, ii + MAX}], "\n=========="];
 ];
```
```
Part 0:
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
==========

Part 1:
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
==========
```

```
Part 2:
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
==========
```

By the closure property, we obtain an annihilating ideal for $\sum_b O_b \cdot c_{n,b}$ = anncii[0]$\cdot \Sigma^{(1)}_{i,n}$. We then compute the singularities.

```
In[•]:= anntotal = DFinitePlus[Sequence @@ Table[annpart[bb], {bb, 0, BB}]];

In[•]:= LeadingExponent[anntotal[[1]]][[1]]

Out[•]= 13

In[•]:= Solve[
         (LeadingCoefficient[anntotal[[1]]] /. {n → n - LeadingExponent[anntotal[[1]]][[1]]}) ==
          0, n, Integers]

Out[•]= {{n → -8}, {n → -7}, {n → -7}, {n → -5}, {n → -3}, {n → -2}, {n → -1}, {n → 0}, {n → 1}}
```

We check that the initial values for $\sum_b O_b \cdot c_{n,b}$ = anncii[0]$\cdot \Sigma^{(1)}_{i,n}$ are all 0, and hence they vanish for all $n \geq i + 2$. **This confirms that $\Sigma^{(1)}_{i,n}$ is annihilated by anncii[0].**

```
In[•]:= Table[Sum[partunderannc0[bb] /. {n → nn, ff → matc}, {bb, 0, BB}],
         {nn, ii + 2, ii + 2 + LeadingExponent[anntotal[[1]]][[1]]}]

        ClearAll[sigma1, hh];
        sigma[nn_] := Sum[mata1[ii, j] * matc[nn, j], {j, 0, nn - 1}];
        sigma1underannc0 = ApplyOreOperator[anncii[0][[1]], hh[n]];
        Table[sigma1underannc0 /. {n → nn, hh → sigma},
         {nn, ii + 2, ii + 2 + LeadingExponent[anntotal[[1]]][[1]]}]

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Note that $a2_{i,j}$ equals Binomial[$-i+3$, 3] when $j = 0$, and 0 when $j \geq 1$ for our choice of i. **Hence, $\Sigma^{(2)}_{i,n} = \sum_{j=0}^{n-1} a2_{i,j}\, c_{n,j}$ = Binomial[$-i + 3$, 3]$* c_{n,0}$ is also annihilated by anncii[0].**

Consequently, $\Sigma_{i,n}$ is annihilated by anncii[0].

Finally, after computing the singularities, we check that $\Sigma_{i,n}$ vanishes for its initial values and hence for all $n \geq i + 2$.

```
In[•]:= LeadingExponent[anncii[0][[1]]][[1]]

Out[•]= 5

In[•]:= Solve[(LeadingCoefficient[anncii[0][[1]]] /.
            {n → n - LeadingExponent[anncii[0][[1]]][[1]]}) == 0, n, Integers]

Out[•]= {{n → -1}, {n → 1}}

In[•]:= Table[Sum[mata[ii, j] * matc[n, j], {j, 0, n - 1}],
         {n, ii + 2, ii + 2 + LeadingExponent[anncii[0][[1]]][[1]]}]

Out[•]= {0, 0, 0, 0, 0, 0}
```

Proof of (H3)

Annihilator for $a_{n-1,j} * c_{n,j}$. Recall that $a_{n-1,j}$ is split into two parts $a1_{n-1,j} + a2_{n-1,j}$. **No need to exe-**

cute the following codes again. Instead, import the data directly.

```
In[•]:= start = CurrentDate[];

ClearAll[n, j];

annH3Smnd1 = DFiniteTimesHyper[annc, mata1[n - 1, j]];
annH3Smnd2 = DFiniteTimesHyper[annc, mata2[n - 1, j]];
Export["annH3Smnd1.txt", {annH3Smnd1}]
Export["annH3Smnd2.txt", {annH3Smnd2}]

Print["Time used: ", CurrentDate[] - start];
Out[•]= annH3Smnd1.txt
Out[•]= annH3Smnd2.txt
Time used: 4.31851 s
```

$a1_{n-1,j} * c_{n,j}$

Import the annihilator for $a1_{n-1,j} * c_{n,j}$.

```
In[•]:= annH3Smnd1 = ToExpression[Import["annH3Smnd1.txt"]];
AnnInfo[annH3Smnd1]
```

ByteCount: 1 211 264

Support: $\{\{S_n^2, S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n S_j^2, S_n S_j, S_j^2, S_n, S_j, 1\}\}$

degree {n, j}: {{23, 23}, {21, 25}, {15, 15}}

Standard Monomials: $\{1, S_j, S_n, S_j^2, S_n S_j\}$

Holonomic Rank: 5

Import the telescoper for $a1_{n-1,j} * c_{n,j}$.

```
In[•]:= annH3CT1 = ToExpression[Import["annH3CT1.txt"]];
In[•]:= AnnInfo[annH3CT1]
```

ByteCount: 6 530 808

Support:
$\{\{S_n^{20}, S_n^{19}, S_n^{18}, S_n^{17}, S_n^{16}, S_n^{15}, S_n^{14}, S_n^{13}, S_n^{12}, S_n^{11}, S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

degree {n}: {{585}}

Standard Monomials:
$\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9, S_n^{10}, S_n^{11}, S_n^{12}, S_n^{13}, S_n^{14}, S_n^{15}, S_n^{16}, S_n^{17}, S_n^{18}, S_n^{19}\}$

Holonomic Rank: 20

```
In[•]:= deltaH3CT1 = ToExpression[Import["deltaH3CT1.txt"]];
In[•]:= ByteCount[deltaH3CT1]
Out[•]= 3 827 584 248
```

Longest digits of the coefficients in the annihilator.

```
In[•]:= Table[Max[IntegerLength[CoefficientList[annH3CT1[[1]][[1]][[jj]][[1]], n]]],
  {jj, 1, LeadingExponent[annH3CT1[[1]]][[1]]}]
Max[
 %]
Out[•]= {1000, 1001, 1002, 1003, 1004, 1005, 1006, 1006, 1007,
 1007, 1008, 1008, 1008, 1008, 1008, 1007, 1007, 1006, 1006, 1004}
```

Out[∘]= 1008

Verify the telescopers for $a1_{n-1,j} * c_{n,j}$. **Note that this step is VERY Memory-consuming, so we executed the command on an Amazon AWS cluster.** To illustrate the process, we may nonrigorously substitute the parameters with specific values.

```
In[∘]:= Timing[
         OreReduce[MapThread[(#1 + (S[j] - 1) ** #2) &, {annH3CT1, deltaH3CT1}], annH3Smnd1]]
Out[∘]= $Aborted
```

```
In[∘]:= subs = {n → 10};
        {annH3CT1subs, deltaH3CT1subs} =
          OrePolynomialSubstitute[#, subs] & /@ {annH3CT1, deltaH3CT1};
        Timing[OreReduce[MapThread[(#1 + (S[j] - 1) ** #2) &, {annH3CT1subs, deltaH3CT1subs}],
          annH3Smnd1, OrePolynomialSubstitute → subs]]
Out[∘]= {45.4531, {0}}
```

```
In[∘]:= subs = {n → 357};
        {annH3CT1subs, deltaH3CT1subs} =
          OrePolynomialSubstitute[#, subs] & /@ {annH3CT1, deltaH3CT1};
        Timing[OreReduce[MapThread[(#1 + (S[j] - 1) ** #2) &, {annH3CT1subs, deltaH3CT1subs}],
          annH3Smnd1, OrePolynomialSubstitute → subs]]
Out[∘]= {47., {0}}
```

$a2_{n-1,j} * c_{n,j}$

Import the annihilator for $a2_{n-1,j} * c_{n,j}$.

```
In[∘]:= annH3Smnd2 = ToExpression[Import["annH3Smnd2.txt"]];
        AnnInfo[annH3Smnd2]
```

ByteCount: 1 060 176
Support: $\{\{S_n^2, S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_j^3, S_n S_j, S_j^2, S_n, S_j, 1\}, \{S_n S_j^2, S_n S_j, S_j^2, S_n, S_j, 1\}\}$
degree {n, j}: {{22, 20}, {19, 22}, {15, 15}}
Standard Monomials: $\{1, S_j, S_n, S_j^2, S_n S_j\}$
Holonomic Rank: 5

Import the telescoper for $a2_{n-1,j} * c_{n,j}$.

```
In[∘]:= annH3CT2 = ToExpression[Import["annH3CT2.txt"]];
```

```
In[∘]:= AnnInfo[annH3CT2]
```

ByteCount: 6 029 816
Support:
$\{\{S_n^{20}, S_n^{19}, S_n^{18}, S_n^{17}, S_n^{16}, S_n^{15}, S_n^{14}, S_n^{13}, S_n^{12}, S_n^{11}, S_n^{10}, S_n^9, S_n^8, S_n^7, S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$
degree {n}: {{552}}
Standard Monomials:
$\{1, S_n, S_n^2, S_n^3, S_n^4, S_n^5, S_n^6, S_n^7, S_n^8, S_n^9, S_n^{10}, S_n^{11}, S_n^{12}, S_n^{13}, S_n^{14}, S_n^{15}, S_n^{16}, S_n^{17}, S_n^{18}, S_n^{19}\}$
Holonomic Rank: 20

```
In[∘]:= deltaH3CT2 = ToExpression[Import["deltaH3CT2.txt"]];
```

```
In[∘]:= ByteCount[deltaH3CT2]
Out[∘]= 5 432 429 512
```

Longest digits of the coefficients in the annihilator.

In[ ]:=
```
Table[Max[IntegerLength[CoefficientList[annH3CT2[[1]][[1]][[jj]][[1]], n]]],
 {jj, 1, LeadingExponent[annH3CT2[[1]]][[1]]}]
Max[
 %]
```

Out[ ]= {971, 972, 973, 974, 974, 976, 977, 977, 978, 978, 978, 978, 979, 979, 978, 978, 978, 977, 976, 975}

Out[ ]= 979

Verify the telescopers for $a2_{n-1,j} * c_{n,j}$. **Note that this step is VERY Memory-consuming, so we executed the command on an Amazon AWS cluster.** To illustrate the process, we may nonrigorously substitute the parameters with specific values.

In[ ]:=
```
Timing[
 OreReduce[MapThread[(#1 + (S[j] - 1) ** #2) &, {annH3CT2, deltaH3CT2}], annH3Smnd2]]
```

Out[ ]= $Aborted

In[ ]:=
```
subs = {n → 10};
{annH3CT2subs, deltaH3CT2subs} =
  OrePolynomialSubstitute[#, subs] & /@ {annH3CT2, deltaH3CT2};
Timing[OreReduce[MapThread[(#1 + (S[j] - 1) ** #2) &, {annH3CT2subs, deltaH3CT2subs}],
  annH3Smnd2, OrePolynomialSubstitute → subs]]
```

Out[ ]= {47.8594, {0}}

In[ ]:=
```
subs = {n → 357};
{annH3CT2subs, deltaH3CT2subs} =
  OrePolynomialSubstitute[#, subs] & /@ {annH3CT2, deltaH3CT2};
Timing[OreReduce[MapThread[(#1 + (S[j] - 1) ** #2) &, {annH3CT2subs, deltaH3CT2subs}],
  annH3Smnd2, OrePolynomialSubstitute → subs]]
```

Out[ ]= {50.4063, {0}}

$a_{n-1,j} * c_{n,j}$

Check that annH3CT1 and annH3CT2 differ by a scalar. So annH3CT1 also annihilates $\sum_j a2_{n-1,j} * c_{n,j}$.

In[ ]:=
```
annH3CT1LeadingCoefficient = LeadingCoefficient[annH3CT1[[1]]];
annH3CT2LeadingCoefficient = LeadingCoefficient[annH3CT2[[1]]];
Scalar = Factor[annH3CT1LeadingCoefficient / annH3CT2LeadingCoefficient]
```

Out[ ]= $n (1+n)^2 (2+n)^2 (3+n)^2 (4+n)^2 (5+n)^2 (6+n)^2 (7+n)^2 (8+n)^2 (9+n)^2 (10+n)^2 (11+n)^2 (12+n)^2 (13+n)^2 (14+n) (15+n) (16+n) (17+n) (18+n) (19+n)$

In[ ]:=
```
annH3CT1CoeffList = OrePolynomialListCoefficients[annH3CT1[[1]]];
annH3CT2CoeffList = OrePolynomialListCoefficients[annH3CT2[[1]]];
```

In[ ]:=
```
Expand[annH3CT1CoeffList] == Expand[Scalar * annH3CT2CoeffList]
```

Out[ ]= True

In[ ]:=
```
annH3CT = annH3CT1;
```

Check initial values.

Check the integer roots of the leading coefficient.

Order of the recurrence for $\sum_j a_{n-1,j} * c_{n,j}$.

In[ ]:=
```
LeadingExponent[annH3CT[[1]]][[1]]
```

Out[∘]= 20

```
In[∘]:= Solve[
  (LeadingCoefficient[annH3CT[[1]]] /. {n → n - LeadingExponent[annH3CT[[1]]][[1]]}) ==
   0, n, Integers]
```

```
Out[∘]= {{n → 1}, {n → 1}, {n → 2}, {n → 2}, {n → 3}, {n → 3}, {n → 4}, {n → 4},
  {n → 5}, {n → 5}, {n → 6}, {n → 6}, {n → 7}, {n → 7}, {n → 8}, {n → 8},
  {n → 9}, {n → 9}, {n → 10}, {n → 10}, {n → 11}, {n → 11}, {n → 12}, {n → 12},
  {n → 13}, {n → 13}, {n → 14}, {n → 14}, {n → 15}, {n → 15}, {n → 16}, {n → 16},
  {n → 17}, {n → 17}, {n → 18}, {n → 18}, {n → 19}, {n → 19}, {n → 20}, {n → 21}}
```

Simplify the quotient prodform[n]/prodform[n-1].

```
In[∘]:= quot = prodform[n] / prodform[n - 1] /.
     prod[f_, {i, a_, n}] :→ (f /. i → n) * prod[f, {i, a, n - 1}];
  quot = FunctionExpand[quot /. prodsimp /. prod → Product]
```

Out[∘]= $\dfrac{\text{Gamma}\left[\frac{3}{4}+\frac{n}{4}\right]\,\text{Gamma}[-1+6\,n]}{\text{Gamma}[5\,n]\,\text{Gamma}\left[-\frac{1}{4}+\frac{5\,n}{4}\right]}$

Verify the quotient prodform[n]/prodform[n-1] also satisfies the recurrence for $\sum_j a_{n-1,j} * c_{n,j}$.

```
In[∘]:= Timing[OreReduce[annH3CT[[1]], Annihilator[quot, S[n]]]]
```

Out[∘]= {1.85938, 0}

Check the first few (more than necessary) initial values.

```
In[∘]:= Table[Sum[mata[n - 1, j] * matc[n, j], {j, 0, n - 1}] == prodform[n] / prodform[n - 1],
   {n, 1, LeadingExponent[annH3CT[[1]]][[1]] + 10}]
```

```
Out[∘]= {True, True, True, True, True, True, True, True, True, True, True, True, True, True, True,
  True, True, True, True, True, True, True, True, True, True, True, True, True, True, True}
```