MATH 3070 – THEORY OF NUMBERS

Homework 1

Due: Thursday, Sep 22, 2022 (in class)

1. Let n be an integer.

(i). Prove that $3 \mid n^3 - n$. (Hint: Consider the three cases $n = 3k$, $3k+1$ and $3k+2$).

(ii). Prove that $5 \mid n^5 - n$.

(iii). Give an example of n such that $4 \nmid n^4 - n$. Write explicitly the values of n and $n^4 - n$.

2. Let $a \geq 2$ and $n \geq 1$ be integers.

(i). Prove that if $a^n + 1$ is prime, then a must be even.

(ii). Prove that if $k \geq 1$ is an odd integer, then for any real x,

$$x^k + 1 = (x+1) \cdot \left(\sum_{i=0}^{k-1} (-1)^i x^i \right).$$

(iii). Use the above relation to show that if $a^n + 1$ is prime, then n is of the form $n = 2^m$ with $m \in \mathbb{N}$, i.e., n has no odd factors other than ± 1.

3. Let $n \geq 2$ be an integer.

(i). Prove that if n is even, then $n^4 + 4^n$ is composite.

(ii). Prove Sophie Germain's identity

$$x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2).$$

(iii). Use Sophie Germain's identity to show that $n^4 + 4^n$ is also composite if $n \geq 2$ is odd.

4. Let a, b be integers with $a \neq 0$.

(i). Prove that $(a, b) = (a, a+b)$.

(ii). Prove that for any integer k, $(a, b) = (a, ka + b)$.