# 10. Integer partitions

## 10.1 Integer partitions

Integer partitions can be seen as a twin sibling of compositions.

**Definition 10.1** An *integer partition* or a *partition* of an integer $n$ is a way of writing $n$ as the sum of a sequence of positive integers, and the order of these summands does *not* matter. We usually denote by $p(n)$ the number of partitions of $n$, and call $p(n)$ the *partition function*.

R Since for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of $n$, the order of these positive integers does not matter, we usually assume that they are in **weakly decreasing** order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$, as a representative. We also often write a partition as $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$.

**Example 10.1** There are five partitions of 4, namely, 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$. Therefore, $p(5) = 4$. ■

**Definition 10.2** Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of $n$, usually written as $\lambda \vdash n$, we call each $\lambda_i$ a *part* of $\lambda$; call $n = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ the *size* of $\lambda$, denoted by $|\lambda|$; and call the number $\ell$ of parts the *length* of $\lambda$, denoted by $\ell(\lambda)$.

R We assume that 0 has an empty partition, written as $\varnothing$, and thus $p(0) = 1$. For the empty partition $\varnothing$, we have $|\varnothing| = 0$ and $\ell(\varnothing) = 0$.

## 10.2 Generating function for partitions

Another convenient way to represent partitions is through the *frequency notation*. Given a partition $\lambda$, for each positive integer $i$, we may count the number $f_i$ of occurrences of $i$ among the parts in $\lambda$, and we call $f_i$ the frequency of $i$. Hence, we may represent $\lambda$ in the frequency notation $1^{f_1}2^{f_2}3^{f_3}\cdots$, and we often omit the integers whose frequency is zero.

**Example 10.2** The partition $6+6+5+3+3+3+2+1+1+1+1+1$ has the frequency notation $6^2 5^1 3^3 2^1 1^5$. ■

R When using the frequency notation, it is necessary to avoid confusion with products of powers.

Taking advantage the frequency notation, it is easy to determine the generating function of $p(n)$.

**Theorem 10.1** Let $p_{\leq N}(n)$ count the number of partitions of $n$ with largest part at most $N$. We have

$$\sum_{n\geq 0} p_{\leq N}(n)q^n = \prod_{k=1}^{N} \frac{1}{1-q^k}. \tag{10.1}$$

*Proof.* We expand the multiplicand

$$\frac{1}{1-q^k} = 1+q^k+q^{2k}+q^{3k}+\cdots = q^{0\cdot k}+q^{1\cdot k}+q^{2\cdot k}+q^{3\cdot k}+\cdots.$$

Hence, each term $q^{f_k\cdot k}$ enumerates the case where the frequency of $k$ is $f_k$ for $f_k$ a nonnegative integer. Further, if we expand the infinite product $\prod_{k=1}^{N}\frac{1}{1-q^k}$, its terms are of the form $q^{f_1\cdot 1+f_2\cdot 2+\cdots+f_N\cdot N}$, corresponding to a unique partition with frequency notation $1^{f_1}2^{f_2}\cdots N^{f_N}$, which also restricts the largest part to be at most $N$. ■

Letting $N\to\infty$, we immediately see that the generating function of $p(n)$ is given by an infinite product.

**Theorem 10.2** We have

$$\sum_{n\geq 0} p(n)q^n = \prod_{k\geq 1} \frac{1}{1-q^k}. \tag{10.2}$$

We may also apply some additional restrictions to the parts.

**Theorem 10.3** For any positive integers $0<a\leq m$, let $p_{a,m}(n)$ count the number of partitions of $n$ with parts congruent to $a$ modulo $m$. We have

$$\sum_{n\geq 0} p_{a,m}(n)q^n = \prod_{k\geq 0} \frac{1}{1-q^{km+a}}. \tag{10.3}$$

*Proof.* Note that

$$\sum_{n\geq 0} p_{a,m}(n)q^n = \prod_{k\geq 0}\left(q^{0\cdot(km+a)}+q^{1\cdot(km+a)}+q^{2\cdot(km+a)}+\cdots\right) = \prod_{k\geq 0}\frac{1}{1-q^{km+a}},$$

as required. ■

**Theorem 10.4** For any positive integer $s$, let $p_{[s]}(n)$ count the number of partitions of $n$ in which each distinct part appears at most $s$ times, i.e., the frequency $f_k\leq s$ for each $k$. We have

$$\sum_{n\geq 0} p_{[s]}(n)q^n = \prod_{k\geq 1} \frac{1-q^{(s+1)k}}{1-q^k}. \tag{10.4}$$

*Proof.* Note that

$$\sum_{n\geq 0} p_{[s]}(n)q^n = \prod_{k\geq 1}\left(q^{0\cdot k}+q^{1\cdot k}+\cdots+q^{s\cdot k}\right) = \prod_{k\geq 1}\frac{(1-q^k)(1+q+\cdots+q^{sk})}{1-q^k} = \prod_{k\geq 1}\frac{1-q^{(s+1)k}}{1-q^k},$$

as required. ■

## 10.3 “Odd partitions” vs “Distinct partitions”

**Definition 10.3** A partition is called an *odd partition* if all its parts are odd integers, and a partition is called an *even partition* if all its parts are even integers. We denote by $p_o(n)$ the number of odd partitions of $n$, and by $p_e(n)$ the number of even partitions of $n$.

Taking $m = 2$, and $a = 1$ and 2, respectively, in Theorem 10.3, we have the following generating function identities.

**Theorem 10.5** We have

$$\sum_{n\geq 0} p_o(n)q^n = \prod_{k\geq 1} \frac{1}{1-q^{2k-1}}, \tag{10.5}$$

$$\sum_{n\geq 0} p_e(n)q^n = \prod_{k\geq 1} \frac{1}{1-q^{2k}}. \tag{10.6}$$

**Definition 10.4** A partition is called a *distinct partition* if all its parts are pairwise distinct. We denote by $p_D(n)$ the number of distinct partitions of $n$.

From the proof of Theorem 10.4 with $s = 1$, the following generating function identity holds true.

**Theorem 10.6** We have

$$\sum_{n\geq 0} p_D(n)q^n = \prod_{k\geq 1} (1+q^k). \tag{10.7}$$

Euler established a well-known result on odd partitions and distinct partitions.

**Theorem 10.7 (Euler).** For $n \geq 0$, we have $p_o(n) = p_D(n)$.

*Proof.* It suffices to show that $p_o(n)$ and $p_D(n)$ have the same generating function:

$$\sum_{n\geq 0} p_o(n)q^n = \prod_{k\geq 1} \frac{1}{1-q^{2k-1}} = \prod_{k\geq 1} \frac{1}{1-q^{2k-1}}\frac{1-q^{2k}}{1-q^{2k}} = \prod_{k\geq 1} \frac{1-q^{2k}}{1-q^k} = \prod_{k\geq 1} (1+q^k) = \sum_{n\geq 0} p_D(n)q^n,$$

as required. ∎

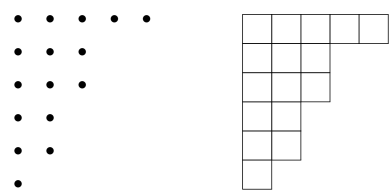
## 10.4 Ferrers diagrams

We may also represent partitions in a graphical way.

**Definition 10.5** A *Ferrers diagram* represents partitions as patterns of dots, with the $n$-th row having the same number of dots as the $n$-th part of the partition. If we replace these dots by squares, the graph is often called a *Young diagram*.

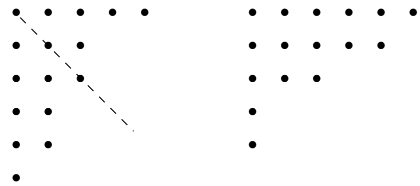
R Ferrers diagrams are named after the British mathematician Norman Macleod Ferrers, and Young diagrams are named after the British mathematician Alfred Young.

■ **Example 10.3** The graphical representations of the partition $5+3+3+2+2+1$ are given as follows — Ferrers diagram (left) and Young diagram (right): ■



**Definition 10.6** Given a partition $\lambda$, its *conjugate partition*, denoted by $\lambda^{\mathsf{T}}$, is the partition whose Ferrers diagram is obtained by fliping the diagram of $\lambda$ along its main diagonal.

**Example 10.4** For the partition $\lambda = 5+3+3+2+2+1$, its conjugate is $\lambda^{\mathsf{T}} = 6+5+3+1+1$. ■



**Theorem 10.8** Let $p(N,n)$ count the number of partitions of $n$ with at most $N$ parts. We have

$$\sum_{n\geq 0} p(N,n)q^n = \prod_{k=1}^{N} \frac{1}{1-q^k}. \tag{10.8}$$

*Proof.* Note that for any partition with at most $N$ parts, its conjugate is a partition with largest part at most $N$. Hence, $p(N,n) = p_{\leq N}(n)$. Recalling Theorem 10.1 gives the desired result. ■

## 10.5 Euler's summations

Note that the above generating functions are represented in the product form. Now, we introduce the *q-Pochhammer symbols* for notational brevity.

**Definition 10.7** Let $q \in \mathbb{C}$ be such that $|q| < 1$. Let $n \in \mathbb{N}$. The *q-Pochhammer symbols* are given by

$$(A;q)_n := \prod_{k=0}^{n-1}(1-Aq^k),$$
$$(A;q)_\infty := \prod_{k\geq 0}(1-Aq^k).$$

We first present refinements of Theorems 10.2 and 10.6.

**Theorem 10.9** Let $\mathscr{P}$ be the set of partitions and $\mathscr{D}$ be the set of distinct partitions. We have

$$\sum_{\lambda\in\mathscr{P}} z^{\ell(\lambda)}q^{|\lambda|} = \frac{1}{(zq;q)_\infty}, \tag{10.9}$$
$$\sum_{\lambda\in\mathscr{D}} z^{\ell(\lambda)}q^{|\lambda|} = (-zq;q)_\infty. \tag{10.10}$$

*Proof.* We have

$$\sum_{\lambda\in\mathscr{P}} z^{\ell(\lambda)}q^{|\lambda|} = \prod_{k\geq 1}\left(1+zq^k+z^2q^{2k}+\cdots\right) = \prod_{k\geq 1}\frac{1}{1-zq^k} = \frac{1}{(zq;q)_\infty}.$$

Similarly,

$$\sum_{\lambda\in\mathscr{D}} z^{\ell(\lambda)}q^{|\lambda|} = \prod_{k\geq 1}\left(1+zq^k\right) = (-zq;q)_\infty,$$

as required. ■

Now, our objective is two important summation formulas due to Euler.

**Theorem 10.10 (Euler's Summations).** We have

$$\sum_{k\geq 0} \frac{z^k q^k}{(q;q)_k} = \frac{1}{(zq;q)_\infty}, \tag{10.11}$$

$$\sum_{k\geq 0} \frac{z^k q^{\frac{k(k+1)}{2}}}{(q;q)_k} = (-zq;q)_\infty. \tag{10.12}$$

*Proof.* For Euler's first summation, we consider partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathscr{P}$ with exactly $k$ parts. Then $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$. Now, we construct a new partition $\lambda' = (\lambda_1', \lambda_2', \dots, \lambda_k')$ with $\lambda_i' = \lambda_i - 1$. Noting that $\lambda_1' \geq \lambda_2' \geq \cdots \geq \lambda_k' \geq 0$, we find that $\lambda'$ is a partition with at most $k$ parts. Since $|\lambda| = |\lambda'| + k$, we have

$$\sum_{\lambda\in\mathscr{P}} z^{\ell(\lambda)} q^{|\lambda|} = \sum_{k\geq 0} z^k q^k \sum_{n\geq 0} p(k,n) q^n = \sum_{k\geq 0} \frac{z^k q^k}{(q;q)_k},$$

where we make use of Theorem 10.8. Recalling (10.9) gives what we want.

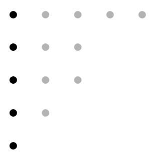
For Euler's second summation, we consider partitions $\pi = (\pi_1, \pi_2, \dots, \pi_k) \in \mathscr{D}$ with exactly $k$ parts. Then $\pi_1 > \pi_2 > \cdots > \pi_k \geq 1$. Now, we construct a new partition $\pi' = (\pi_1', \pi_2', \dots, \pi_k')$ with $\pi_i' = \pi_i - (k+1-i)$. Noting that $\pi_1' \geq \pi_2' \geq \cdots \geq \pi_k' \geq 0$, we find that $\pi'$ is a partition with at most $k$ parts. Since $|\pi| = |\pi'| + (1+2+\cdots+k) = |\pi'| + \frac{k(k+1)}{2}$, we have

$$\sum_{\pi\in\mathscr{D}} z^{\ell(\pi)} q^{|\pi|} = \sum_{k\geq 0} z^k q^{\frac{k(k+1)}{2}} \sum_{n\geq 0} p(k,n) q^n = \sum_{k\geq 0} \frac{z^k q^{\frac{k(k+1)}{2}}}{(q;q)_k},$$

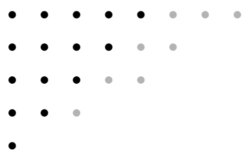
where we also use Theorem 10.8. Recalling (10.10) implies the desired result. ■

R The above proof can also be understood graphically.

Euler's first sum:



Euler's second sum:

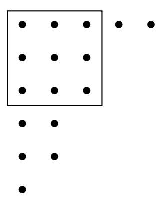


## 10.6 Durfee squares

From the Ferrers diagram of a partition, another important concept can be introduced.

**Definition 10.8** Given a partition, its *Durfee square* is the largest square contained in its Ferrers diagram.

R Durfee squares are named after the American mathematician William Pitt Durfee, a student of James Joseph Sylvester.



**Example 10.5** The partition $5+3+3+2+2+1$ has a Durfee square of size 3, as shown in the Ferrers diagram. ■

**Theorem 10.11** We have

$$\sum_{k\geq 0} \frac{q^{k^2}}{(q;q)_k^2} = \frac{1}{(q;q)_\infty}. \tag{10.13}$$

*Proof.* We consider partitions $\lambda$ whose Durfee square is of size $k$. Note that below the Durfee square, we have a partition $\mu$ with largest part at most $k$; and that to the right of the Durfee square, we have a partition $\nu$ with at most $k$ parts. Since $|\lambda| = |\mu| + |\nu| + k^2$ where $k^2$ is contributed by the Durfee square, we have

$$\sum_{\lambda\in\mathscr{P}} q^{|\lambda|} = \sum_{k\geq 0} q^{k^2} \sum_{n\geq 0} p_{\leq k}(n) q^n \sum_{n\geq 0} p(k,n) q^n = \sum_{k\geq 0} \frac{q^{k^2}}{(q;q)_k^2},$$

where we use Theorems 10.1 and 10.8. The desired identity follows from Theorem 10.2. ∎