9. Generating functions

9.1 Generating functions

In the previous lecture, we have shown the existence of a representation as the sum of four squares for each nonnegative integer n. Now a natural question is how many such representations do we have? Is there a formula, or at least a nice way, to characterize the number of such representations for each n?

In general, for $\{a_n\}_{n\geq 0}$ a sequence of numbers, not necessarily integers, we want to find a clothesline on which we hang up $\{a_n\}$ for display.

Definition 9.1 Let $\{a_n\}_{n\geq 0}$ be a sequence of numbers. The the power series

$$\sum_{n\geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

is called the *generating function* of $\{a_n\}$.

R Since we are considering power series, a natural question is their radii of convergence. However, this question is uaually not very interesting for generating functions, and in many cases we only treat these power series as *formal* power series. However, there are still occassions that the radii of convergence should be taken into account, especially when analytic techniques are applied. For instance, when we want to make use of Cauchy's integral formula to recover the coefficients a_n from its generating function $A(x) = \sum_{n\geq 0} a_n x^n$:

$$a_n = \frac{1}{2\pi i} \oint \frac{A(x)}{x^{n+1}} dx,$$

we must be careful about the convergence conditions when choosing the contour.

9.2 Formal power series

Definition 9.2 A *formal power series* is an expression of the form

$$a_0 + a_1 x + a_2 x^2 + \cdots,$$

where the sequence $\{a_n\}_{n\geq 0}$ is called the *sequence of coefficients*.

We say two series $A(x)=\sum_{n\geq 0} a_n x^n$ and $B(x)=\sum_{n\geq 0} b_n x^n$ are *equal* if $a_n=b_n$ for all $n\geq 0$. We can also define the usual operations for formal power series:

- *Addition/Subtraction*:

$$\sum_{n\geq 0} a_n x^n \pm \sum_{n\geq 0} b_n x^n = \sum_{n\geq 0} (a_n \pm b_n) x^n;$$

- *Multiplication* by the Cauchy product rule:

$$\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0} c_n x^n, \quad \text{where } c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

To determine if division works, we need to check if a series has a *reciprocal*.

Definition 9.3 Given a formal power series $\sum_{n\geq 0} a_n x^n$, we say a series $\sum_{n\geq 0} b_n x^n$ is the *reciprocal* of $\sum_{n\geq 0} a_n x^n$ if

$$\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right) = 1.$$

Theorem 9.1 A formal power series $A(x)=\sum_{n\geq 0} a_n x^n$ has a reciprocal if and only if $a_0 \neq 0$. In that case, the reciprocal is unique.

Proof. (i). If $A(x)$ has a reciprocal, say $B(x)=\sum_{n\geq 0} b_n x^n$. Then $A(x)B(x)=1$. Hence, $a_0 b_0 = 1$, which implies that $a_0 \neq 0$. Further, b_0 is uniquely given by $1/a_0$. Also, for $n\geq 1$, we have $0=\sum_{k=0}^{n} a_k b_{n-k}$. Therefore,

$$b_n = -\frac{1}{a_0}\sum_{k=1}^{n} a_k b_{n-k}.$$

By induction, the b_n's are uniquely determined.

(ii). If $a_n \neq 0$, we choose $b_0 = 1/a_0$, and iteratively define $b_n = -\frac{1}{a_0}\sum_{k=1}^{n} a_k b_{n-k}$. Then we get a series $B(x)=\sum_{n\geq 0} b_n x^n$. It is straightforward to verify that $A(x)B(x)=1$, and hence $B(x)$ is a reciprocal of $A(x)$. ■

Example 9.1 We have

$$(1-x)(1+x+x^2+\cdots)=1.$$

Hence, the reciprocal of $1-x$ is given by $1+x+x^2+\cdots$, written as

$$\frac{1}{1-x} = 1+x+x^2+\cdots.$$

This is exactly identical to what is obtained by applying the Taylor expansion to $\frac{1}{1-x}$.

Definition 9.4 Let $A(x)=\sum_{n\geq 0} a_n x^n$ be a formal power series. Its *derivative* is the series

$$A'(x) = \sum_{n\geq 1} n a_n x^{n-1}.$$

Example 9.2 We know that

$$e^x = \sum_{n\geq 0} \frac{x^n}{n!}.$$

Now,

$$\left(\sum_{n\geq 0} \frac{x^n}{n!}\right)' = \sum_{n\geq 1} \frac{n x^{n-1}}{n!} = \sum_{n\geq 1} \frac{x^{n-1}}{(n-1)!} = e^x.$$

This is exactly identical to $(e^x)'=e^x$.

9.3 Fibonacci numbers

The *Fibonacci numbers* are named after the Italian mathematician Leonardo of Pisa, later known as Fibonacci, for his famous "*Rabbit Puzzle*" in his 1202 book *Liber Abaci*:

> *Assume that we have a pair of fictional rabbits, and they*
>
> (i) *produce a new pair of rabbits every month, starting from the second month that they are alive;*
>
> (ii) *and the new generations always repeat the trajectory of their parents' life.*
>
> *If rabbits never die and continue breeding forever, how many pairs will there be in one year?*

Assume that there are F_n pairs of rabbits after n months, starting with $F_0 = 0$ and $F_1 = 1$. Now, for F_n with $n \geq 2$, the rabbits are from the alive ones of the previous month, F_{n-1} pairs in total, and the newly born rabbits produced by those of at least two-month-old, F_{n-2} pairs in total. Therefore, for $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}. \tag{9.1}$$

Theorem 9.2 We have

$$\sum_{n\geq 0} F_n x^n = \frac{x}{1-x-x^2}. \tag{9.2}$$

Proof. We multiply (9.1) by x^n, and then sum over $n \geq 2$. Then

$$\sum_{n\geq 2} F_n x^n = \sum_{n\geq 2} (F_{n-1} + F_{n-2}) x^n = x \sum_{n\geq 2} F_{n-1} x^{n-1} + x^2 \sum_{n\geq 2} F_{n-2} x^{n-2} = x \sum_{n\geq 1} F_n x^n + x^2 \sum_{n\geq 0} F_n x^n.$$

Let $f = \sum_{n\geq 2} F_n x^n$. We have

$$f - (0 + x) = x(f - 0) + x^2 f,$$

or

$$(1 - x - x^2) f = x.$$

This gives the desired result. ■

Can we find an explicit formula for F_n?

Theorem 9.3 For $n \geq 0$,

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right) \tag{9.3}$$

Proof. Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then $(1 - x - x^2) = (1 - \alpha x)(1 - \beta x)$. Therefore,

$$\begin{aligned} \frac{x}{1-x-x^2} &= \frac{x}{(1-\alpha x)(1-\beta x)} = \frac{1}{\alpha - \beta} \left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x} \right) \\ &= \frac{1}{\alpha - \beta} \left(\sum_{n\geq 0} \alpha^n x^n - \sum_{n\geq 0} \beta^n x^n \right). \end{aligned}$$

By equating the coefficient of x^n, we have

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

which is exactly as desired. ■

In general, we may consider the sequence $\{G_n\}_{n\geq 0}$ with $G_0 = a$, $G_1 = b$, and for $n \geq 2$, $G_n = sG_{n-1} + tG_{n-2}$ where a, b, s and t are fixed.

Theorem 9.4 We have

$$\sum_{n\geq 0} G_n x^n = \frac{a + bx - asx}{1 - sx - tx^2}. \tag{9.4}$$

Proof. Note that

$$\sum_{n\geq 2} G_n x^n = sx \sum_{n\geq 1} G_n x^n + tx^2 \sum_{n\geq 0} G_n x^n.$$

Thus,

$$\sum_{n\geq 0} G_n x^n - (a + bx) = sx \left(\sum_{n\geq 0} G_n x^n - a \right) + tx^2 \sum_{n\geq 0} G_n x^n,$$

yielding the desired result. ■

For example, the *Lucas numbers* L_n, which were introduced by the Frence mathematician François Lucas, are given by $L_0 = 2$, $L_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

Theorem 9.5 We have

$$\sum_{n\geq 0} L_n x^n = \frac{2 - x}{1 - x - x^2}. \tag{9.5}$$

In particular, for $n \geq 0$,

$$L_n = \left(\frac{1+\sqrt{5}}{2} \right)^n + \left(\frac{1-\sqrt{5}}{2} \right)^n. \tag{9.6}$$

Proof. The first part is the $(a, b, s, t) = (2, 1, 1, 1)$ case of Theorem 9.4. For the second part, we still write $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then

$$\frac{2 - x}{1 - x - x^2} = \frac{1}{1 - \alpha x} + \frac{1}{1 - \beta x} = \sum_{n\geq 0} \alpha^n x^n + \sum_{n\geq 0} \beta^n x^n.$$

Equating the coefficient of x^n implies the desired result. ■

9.4 Compositions

Generating functions are of significant use in combinatorics. Here, we will take compositions as an example.

Definition 9.5 A *composition* of an integer n is a way of writing n as the sum of a sequence of positive integers, and the order of these summands matters.

■ **Example 9.3** There are four compositions of 3, namely, 3, $2+1$, $1+2$ and $1+1+1$. ■

Theorem 9.6 There are 2^{n-1} compositions of n.

Proof. We represent the integer n by n nodes in a row. Then there are $n-1$ gaps between consecutive nodes. Now, let us choose to place a stick at each gap or not, and there are 2^{n-1} choices. Each choice will induce a unique composition of n by counting the number of nodes between each consecutive pair of sticks while we assume that there are two invisible sticks at the two ends. Hence, there are 2^{n-1} compositions of n.

$$\bullet \quad \bullet \;|\; \bullet \quad \bullet \quad \bullet \;|\; \bullet \quad \bullet \;|\; \bullet \;|\; \bullet$$

For instance, the above diagram gives $2+3+2+1+1$, which is a composition of 9. ■

Is it possible to avoid such a combinatorial argument?

Theorem 9.7 Let $c(k,n)$ count the number of compositions of n into k parts. Then

$$\sum_{n\geq 1} c(k,n)x^n = \left(\frac{x}{1-x}\right)^k. \tag{9.7}$$

Proof. Let us consider the product

$$(x+x^2+\cdots)^k = (x+x^2+\cdots)(x+x^2+\cdots)\cdots(x+x^2+\cdots),$$

where there are k multiplicands. If we expand this product, then the terms are of the form $x^{n_1+n_2+\cdots+n_k} =: x^n$ where each x^{n_i} comes from the i-th multiplicand. Also, this term corresponds to a unique composition of n, given by $n_1+n_2+\cdots+n_k$, and there are exactly k parts in this composition. Hence,

$$\sum_{n\geq 1} c(k,n)x^n = (x+x^2+\cdots)^k = \left(\frac{x}{1-x}\right)^k,$$

as required. ■

Theorem 9.8 Let $c(n)$ count the number of compositions of n. Then

$$\sum_{n\geq 1} c(n)x^n = \frac{x}{1-2x}. \tag{9.8}$$

In particular, $c(n) = 2^{n-1}$.

Proof. For the first part, we deduce from Theorem 9.7 that

$$\sum_{n\geq 1} c(n)x^n = \sum_{k\geq 1}\sum_{n\geq 1} c(k,n)x^n = \sum_{k\geq 1}\left(\frac{x}{1-x}\right)^k = \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} = \frac{x}{1-2x}.$$

Further, $\frac{x}{1-2x} = \sum_{n\geq 1} 2^{n-1}x^n$. By equating the coefficient of x^n, we arrive at the second part. ■